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HANS FREUDENTHAL'S REALISTIC MATHEMATICAL THEORY: DIDACTICS AND RESEARCH PARADIGMS

RESEARCH BOOK

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Introduction

Freudenthal was an enthusiastic advocate of reform in traditional mathematics education. His extensive work as a founder and active participant in groups such as the International Group on Psychology and Mathematics Education (PME) and the International Commission for the Study and Improvement of the Teaching of Mathematics (CIEAEM) contributed to his popularity. In these forums he expressed his opposition to the pedagogical and didactic approaches that prevailed in the mid-twentieth century, such as operational goal theory, structured assessment tests, standardized educational research, and the direct application of Piaget's structuralism and constructivism in the classroom.

He also criticized the separation between educational research, curriculum development and teaching practice, as well as the introduction of "modern" mathematics into schools. Freudenthal's publications on Mathematics Education spanned many years, during which he collaborated with other members of the Institute for the Development of Mathematics Education (IOWO), which he founded in 1970 at Utrecht University. Today the institute is known as the Freudenthal Institute. Together, the members of the group worked in schools, alongside regular teachers, studying students' informal knowledge and finding ways to connect it to proposed activities and models. They designed and tested sequences, continually improving them based on analysis of their implementation. This work laid the foundation for the current approach known as Realistic Mathematics Education (RME).

Hans Freudenthal, a mathematician and educator of German descent, earned his doctorate at the University of Berlin. However, due to his Jewish heritage, he was forced to emigrate from Germany during the rise of the Nazi regime. He found refuge in the

Netherlands, where he continued his academic career and developed his pedagogical theories. Unfortunately, he had to remain in hiding during the years of World War II. Freudenthal believed that the learning process should be based on situations that require organization.

He criticized Piaget for attempting to impose psychological development on the system of categories used by mathematicians, using mathematical terminology with different meanings. Based on his own experiences, Freudenthal argued that learning was more closely related to linguistic development than to cognitive development. He was concerned with how Piaget's work influenced teaching methodologists to translate research findings into instructional guidelines for mathematics education, transforming an epistemological theory into a pedagogical theory.

He held discussions with Chevallart about his theory of transposition, which he believed was based on the expert knowledge of mathematicians. Freudenthal argued that mathematics taught in schools should not reflect any interpretation of philosophical or scientific ideas, unless they were from a much earlier period.

Freudenthal's opposition to the prevailing psychology, pedagogy, and didactics of the time was founded. It was rooted in his deep knowledge of the mathematical discipline, his passion for teaching it, and his first-hand experience in the classroom. He questioned the artificial nature of the educational goals and learning domains proposed by Bloom, arguing that they had a negative impact on both school tests and developmental testing. He accused Bloom of conceiving of learning as a process in which knowledge is simply poured into students' heads. Similarly, he disagreed with Gagné's view that learning is a continuous process that progresses from simple to complex structures.

Freudenthal believed that learning involved sudden leaps of reinvention, demonstrated by students experiencing "aha" moments, developing shortcuts in their strategies, shifting their perspectives, and using models of varying levels of formalization. He argued that learning actually moves from rich, complex structures of the real world to the more general, abstract, and formal structures of mathematics. Although Freudenthal's references to non-mathematical authors were limited, he acknowledged influences from Decroly, whose interests aligned with his own theory of learning mathematics in real-life contexts, and Dewey, with whom he saw similarities in the idea of guided reinvention. He was also inspired by Pierre and Dina Van Hiele, incorporating their levels of mathematization into his work on the development of geometric thinking and its didactics. He was also influenced by the phenomenological pedagogy of Lagenveld, the intuitive didactics of Castelnuovo E., the progressive education of Petersen, Kry Van Perreren and the sociocultural theories of Eastern Europe.

Realistic Mathematics Education, as presented in this book, does not claim to be a comprehensive learning theory like constructivism, but is a comprehensive philosophy (according to Freudenthal) that is implemented through a set of teaching theories specific to mathematical subjects. The central ideas of this approach are the following: - Mathematics is considered a human activity (what Freudenthal calls mathematization) and therefore should be accessible to all. - The development of mathematical understanding occurs in different stages where contexts and models play an important role.

This development is facilitated through the process of guided reinvention, within a diverse cognitive environment. - From a curricular perspective, the guided reinvention of mathematics as a mathematization activity requires the use of didactic phenomenology as a research methodology.

It is about searching for contexts and situations that generate the need for mathematical organization. The history of mathematics and students' spontaneous mathematical inventions and productions serve as primary sources for this search. These concepts, commonly known as the Principles of Realistic Mathematics Education, are explained in more detail below: The Activity Principle emphasizes that mathematics should be seen as a human activity that can be accessed and learned through active participation.

According to Freudenthal, teaching the process of mathematical activity is more important than teaching the end result. The focus should not be solely on learning algorithms or concepts, but on the process of algorithmization, algebrization, abstraction, formalization, and structuring. According to this principle, mathematics should be accessible to all students, recognizing that not all need to pursue careers in mathematics. The goal is for students to develop mathematical and critical thinking skills to apply to everyday problems.

The emphasis is on providing access to knowledge, skills and dispositions through real-life situations, uncovering the hidden processes within mathematical products. Freudenthal draws inspiration from the activities of mathematicians, whether in pure or applied mathematics, who are engaged in problem solving and problem-solving and in organizing content related to mathematical concepts and real-world information.

Chapter 1

Mathematician and Theorist: Hans Freudenthal

Throughout his professional career, Hans Freudenthal's perspectives on educational reform diverged from all contemporary approaches. He questioned the "new" mathematics, operational objectives, rigid assessment methods, standardized empirical quantitative research, and strict divisions between curriculum research, development, and implementation. Interestingly, while his ideas were initially seen as rebellious, they have now gained widespread acceptance. This suggests the important role that Hans Freudenthal played not only in mathematics education but also in curriculum theory and methodological research.

In addition to his reputation as a mathematical researcher, Freudenthal also delved into the educational and psychological traditions of Europe and the United States, making his own contributions to mathematics education. Today, he is widely recognized as one of the most influential mathematics educators of his time. In this paper we aim to highlight some of Freudenthal's ideas, although it is impossible to cover them all. We will focus on pedagogy and curriculum theory, exploring aspects of Freudenthal's work and the theories that are relevant from these perspectives.

There are notable differences between the curriculum theory developed by educators in the United States and Europe, despite arguments that they address similar issues. These differences arise from fundamental disparities in cultural, philosophical, and institutional backgrounds. In Europe, pedagogical theory includes the concept of Didactics, which is considered a form of humanities.

This perspective is based on the practice of education, focusing on realist education and the phenomenological theory of Bildung, which encompasses the formation of the individual's personality. It goes beyond the mere transmission of knowledge and also emphasizes the development of norms, values, and skills necessary to be a "good" citizen or part of an intellectual elite. On the other hand, Ausbildung refers to vocational and professional training. Didactics in this context is primarily concerned with theories about the purpose and content of education and instruction.

In the Netherlands, didactics is influenced by the phenomenological pedagogy of the Geisteswissenschaftliche, as exemplified by the work of Langeveld at Utrecht University in 1965. Although this perspective lost importance in the 1960s and 1970s, leading to the gradual replacement of a general didactic perspective with formal models of learning and teaching popularized by American educational psychologists such as Robert Glaser, Robert de Cecco and Benjamin Bloom. Despite this shift, the content of didactics developed in faculties and institutes of mathematics and educational sciences was not completely overshadowed by this movement.

Despite never mentioning students like Wolfgang Klafki, Freudenthal's questions about what should be taught in school subjects, for what purpose, and for whom are similar to those posed by Klafki. Freudenthal's belief in "mathematics as a human activity" can be seen as a representation of a Geisteswissenschaftliche, a phenomenological theory of mathematics education that focuses on the practical aspects of teaching and education rather than simply transmitting pre-existing mathematical knowledge. Some of Freudenthal's main ideas, such as "reinvention" and his critique of the "anti-didactic inversion" of traditional deductive instruction, may have been influenced by the progressive education and pedagogical reform movement. Figures such as Peter Petersen and Maria Montessori played a role in shaping these ideas.

According to Freudenthal, curriculum theory is not a fixed set of theories and purposes but is process-dependent. The term "curriculum" is often used in conjunction with change or development, such as curriculum development or research development. For Freudenthal, curriculum theory is a practical endeavour that can lead to the emergence of new theoretical ideas. He believes that curriculum development should not be led by academic leaders but should involve collaboration between teachers and students in schools. Similar ideas are shared by Schwab, who advocates curriculum as "practice" and challenges the dominant curriculum theory of his time. As a result, there are similarities between certain branches of the Anglo-Saxon approach to curriculum theory and Freudenthal's understanding of curriculum (Gravemeijer & Terwel, 2000).

However, in Freudenthal's writings there is often a negative connotation associated with the term 'curriculum'. He describes the dominant Anglo-Saxon curriculum movement as a behaviorist and top-down theory, referring to it as 'boxology'. In contrast, Freudenthal presents his own view of curriculum as a process, which he calls educational development. Whereas curriculum development focuses on the creation of curriculum materials, Freudenthal seeks to go a step further by promoting changes in classroom teaching through educational development.

Educational development therefore goes beyond instructional design and encompasses comprehensive strategic innovation. This innovation is based on an explicit educational philosophy and involves the development of diverse materials as part of the overall strategy. Research plays a crucial role in driving this entire process, aligning well with the pedagogical tradition. Specifically, qualitative and interpretive research is employed, drawing on teaching experiences in individual classes. Dialogue between researchers, curriculum developers and teachers are central to this approach.

Mathematics: human activity

Freudenthal was known for his opposition to the "new mathematics" of the 1960s, which was based on modern mathematics and set theory. He believed in traditional pedagogy and criticized the new approach because he believed it neglected what should be taught and how it should be taught. He recognized that mathematics is characterized by its generality and wide applicability, but he also saw abstraction as a problem in teaching. While abstract mathematics is flexible in an objective sense, it may not be useful to people who cannot apply this flexibility to their own lives. Freudenthal argued that mathematics should be taught as a useful tool, but not simply by teaching mathematical concepts and then applying them. He believed that the order of teaching was important and that mathematics should be taught by mathematizing. This approach emphasizes the process of doing mathematics rather than focusing solely on the end result. In traditional mathematics education, the starting point is often the result of the mathematical activity of others, which Freudenthal saw as an anti-didactic reversal. He believed that teaching should begin with the activity itself and not the end result.

The pursuit of mathematics involves both problem solving and the establishment of a structured discipline. To effectively solve real-world problems, they need to be organized and addressed using mathematical patterns. Likewise, mathematics itself requires organization, whether by organizing new or existing results, whether one's own or those of others, to enhance understanding. This may involve exploring new ideas, examining broader contexts, or applying an axiomatic approach, as Freudenthal suggested in 1971.

Freudenthal's approach to mathematization includes both "subjects of reality" and "mathematical subjects," encompassing both applied mathematics and pure mathematics.

This sets him apart from other mathematics educators who also emphasize mathematical activity but base their discourse on the discourse of pure mathematics researchers. Freudenthal's description of mathematical activity as a model for mathematics education differs from the above in two respects:

- First, it incorporates applied mathematics or the process of using mathematics to solve real-world problems.
- Second, it shifts the focus from the structure of the activity to the activity itself and its results.
- Furthermore, the concept of "discourse" refers to a social practice in which the act of mathematizing gives significant importance to mental engagement.

Freudenthal's comprehensive definition of mathematics as a human endeavor aligns most effectively with a practical discourse, such as that found in applied mathematics. In this type of discourse, there is a greater emphasis on effectiveness and efficiency, and less focus on speculative guesswork without a clear goal. He employs the term "mathematize" comprehensively, encompassing both the organization and application of mathematical principles. By selecting the word "organize," Freudenthal conveys that mathematizing involves more than simply translating concepts into a structured system of symbols. Furthermore, the act of organizing the subject matter itself should lead to the development of a symbolic representation.

Precision is also a key aspect of mathematization. Mathematics is known for its precision and accuracy, and by applying mathematical reasoning, we can ensure that our solutions are accurate and error-free. We use mathematical tools, such as formulas, equations, and calculations, to arrive at accurate and reliable results. By mathematizing, we can avoid misconceptions and misinterpretations, leading to a more accurate mathematical understanding.

When we talk about making something "more mathematically," we are referring to the process of applying mathematical principles and concepts in a way that emphasizes generality, certainty, precision, and brevity. Generality refers to the ability to apply mathematical ideas in diverse contexts and situations. By mathematizing, we can recognize patterns, identify relationships, and make connections that extend beyond specific examples. This allows us to solve a wide range of problems and understand the underlying principles that govern them. Certainty is another characteristic of mathematization.

When approaching a problem mathematically, we strive for logical reasoning and evidence-based solutions. We rely on the rules and principles of mathematics to guide our thinking and ensure our conclusions are dependable and well-founded. Mathematics has a unique language and symbolism that allows us to express complex ideas and concepts concisely. When mathematizing, we aim to use this language effectively, using symbols, notation, and concise explanations to communicate mathematical ideas efficiently. This allows for clearer communication and a more streamlined approach to problem solving:

- By generality: generalizations (observation of analogies, classifications, structures)
- In order to establish certainty, it is crucial to engage in a process of reflection, justification and testing. This can be achieved by employing a systematic approach, which involves developing and testing conjectures, hypotheses or theories. By thoroughly examining the evidence and subjecting it to rigorous scrutiny, the validity and reliability of the conclusions drawn can be determined.

By emphasizing the importance of critical thinking and logical reasoning, this systematic approach ensures that certainty is not simply assumed but substantiated through sound and rigorous analysis.

- To ensure accuracy, it is important to use multiple methods, such as modeling, symbolizing, and defining, to limit interpretations and assess the validity of information.
- To maintain conciseness, it is essential to symbolize and schematize, which involves creating standardized procedures and notations.

When viewed from this perspective, the act of mathematizing mathematical objects and the act of mathematizing real-world issues have similar characteristics. This is a crucial concept for Freudenthal, as he suggests that children's mathematics education should focus on the application of mathematical principles to everyday situations. Children are not able to mathematize mathematics itself, as they have no direct experience with mathematical objects. Similarly, when students mathematize real-world disciplinary objects, they become more familiar with using mathematical approaches to solve problems in their daily lives. This also relates to Freudenthal's idea of "problem-finding," which involves having a mathematical mindset that understands the strengths and limitations of using mathematics in different situations.

The notion of “mathematizing reality” is a central aspect of the concept of “mathematics for all.” Freudenthal recognizes that not all students will become mathematicians in the future but emphasizes that the mathematics they learn must be applicable to everyday problem solving. It is therefore important to prioritize teaching students how to approach problem solving using mathematical methods. This objective can be combined with the goal of students applying mathematical concepts to situations that are relevant to their own experiences. From this perspective, it is not surprising that Freudenthal strongly criticizes the concept of didactic transposition, proposed by Chevallard (1985), which relies on the expert knowledge of mathematicians. Freudenthal argues that mathematics taught in schools should not be a mere translation of philosophical or scientific ideas, unless they are from a much earlier time (Gravemeijer & Terwel, 2000).

Keitel (1987) argues that the main goal is to develop a mathematics curriculum that is accessible to all individuals while retaining the essence of mathematics itself. To achieve this, he suggests that teachers should sometimes move away from real-world problems and focus on the concepts, structures and systems that have been established and tested within mathematical science. Building on Freudenthal's concept of mathematization, the idea of horizontal and vertical mathematization is introduced.

Horizontal mathematization involves transforming a contextual problem into a mathematical problem, while vertical mathematization involves taking the mathematical discipline to a higher level. Vertical mathematization can be fostered by presenting problems that have mathematical solutions at various levels of complexity. Freudenthal (1991) describes this distinction by explaining that horizontal mathematization bridges the gap between the real world and the realm of symbols.

In the real world, individuals live, act, and experience various emotions, while in the symbolic world symbols are created, manipulated, and understood through mechanical, integral, and reflexive processes. The real world represents what is perceived as reality, while the symbolic world represents abstraction. However, the boundaries between these two worlds are not clearly defined and can fluctuate. Freudenthal emphasizes that the distinction between horizontal and vertical mathematization is not rigid, as the perception of reality varies from person to person. He defines reality as a combination of interpretation and sensory experience, suggesting that mathematics can also be part of an individual's reality. The concept of reality and what is considered common sense is not fixed but is influenced by personal learning processes. Therefore, Freudenthal's statement that "mathematics begins and remains in reality" should be interpreted as an acknowledgement of the dynamic nature of reality and its relationship to mathematics.

In Freudenthal's perspective, the concepts of "common sense" and "reality" are subjective and depend on the individual's point of view. This means that the distinction between vertical and horizontal mathematization must also be assessed from the individual's perspective. Whether a specific mathematical activity is considered "vertical" or "horizontal" depends on the nature of the activity and the person's understanding of mathematics. For example, a symbolic activity may be routine for a student, categorizing it as horizontal mathematization.

However, if the same symbolic activity involves a new invention for another student, it would be considered vertical mathematization. The latter is most evident when a student replaces his or her method of solving or way of describing with a more sophisticated, organized, and mathematical approach. These changes can be fostered by reflecting on the methods of solving and deepening understanding. Engaging in whole-class discussions that explore different solution methods, interpretations, and ideas can contribute to these changes. Thus, during such discussions, students may discover alternative solution methods that are more advantageous than the current ones. This highlights the importance of dialogue in mathematization, emphasizing that it is not solely an individual activity.

Likewise, Freudenthal also emphasizes the importance of group work in mathematics education. He first introduced the concept of small group learning in 1945 and later advocated mathematics education in diverse groups. According to Freudenthal, both hardworking and lazy students can benefit from collaborative learning. Surprisingly, reviewing his works from the 1940s onwards, Freudenthal found that he had consistently advocated cooperative learning in small and diverse groups.

The Criticism

Freudenthal's reputation is not only based on his own theoretical ideas, but also on his criticism of "traditional" research. In the educational research community in the Netherlands, he encountered strong opposition for his stance against those who relied on an empiricist methodology and complicated statistical analyses. Drawing on his experience as a mathematician, Freudenthal skillfully exposed the significant shortcomings in the application of mathematics and statistics in numerous cases of supposedly "high" empirical research.

Freudenthal's stance against much of educational research stems from his belief that interruptions in the learning process are crucial. These disruptions can be seen as shortcuts or opportunities to gain different perspectives. According to Freudenthal, it is through these disruptions that it can be determined whether a student has reached a certain level of understanding. To identify these disruptions, individual students must be closely monitored. This approach ignores the importance of groups and the removal of individual disruptions. Furthermore, the focus should be on observing the learning process rather than testing the achievement of learning objectives. Overall, Freudenthal argued that traditional research methods could not adequately address educational questions about the purpose and target audience of a particular topic.

Freudenthal expressed further concerns and objections towards the testing movement and offered a second round of criticism. His skepticism revolved around the methods employed in testing and he harshly criticized the detrimental impact that examinations and testing techniques had on the field of education. The crux of his criticism centered on the lack of understanding of the subject matter being tested and the excessive emphasis placed on reliability, while ignoring the importance of validity. Freudenthal clearly did not share the same positive outlook and enthusiasm as the proponents of objective testing.

In a broader sense, Freudenthal's critique of educational research focuses on methodologists who possess extensive knowledge about research methods but lack an understanding of education itself. He vehemently opposes the division between content and form, arguing that this approach results in empty models that require experts to fill them with educational substance. These models do not consider whether the content actually aligns with educational principles (Gravemeijer & Terwel, 2000). Furthermore, he voices comparable objections toward comprehensive educational theories.

According to Freudenthal, general educational theories do not align with the specific needs of mathematics education and may even be detrimental to the type of education they are intended to support. He specifically criticizes the educational theories proposed by Bloom, Gagné, and Piaget. Freudenthal argues that Bloom's Taxonomy of Educational Objectives is not suitable for mathematics education because it focuses on classification rather than the active process of structuring reality. He believes that students gain control over reality through this structuring process, and the artificial categories of Bloom's Taxonomy have a negative impact on both school and developmental testing.

He also rejects Bloom's mastery learning strategy, accusing it of treating learning as a passive process of knowledge transfer. Similarly, he disagrees with Gagné's concept of task analysis, as it does not align with his view of mathematics as a human activity.

Freudenthal questions whether mathematics is really so different from other disciplines and expresses the wish that someone with a background in both mathematics and psychology would bridge this gap.

While Gagné sees learning as a continuous progression from simple to complex structures, Freudenthal sees it as a discontinuous process from the rich and complex structures of everyday life to the abstract structures of symbolic mathematics. He believes that the starting points for learning should be situations that require organization and that learners should develop their own categories based on their needs.

Freudenthal also criticizes Piaget for his approach to mathematics and his experiments. However, what he is more concerned about is how Piaget's work influences teaching methodologies by basing their practices on theories they have learned from a psychologist. He argues that these methodologies often misinterpret or misinterpret Piaget's mathematical presuppositions rather than being based on the actual findings of his experiments.

In his work, Freudenthal delves deeper into the concept of constructivism and offers both criticism and support for this epistemology. While he criticizes constructivist epistemology as an observer, he argues that his own perspective as an actor aligns with this epistemology. Specifically, he views mathematics from the perspective of a practicing mathematician and characterizes it as a well-developed form of common sense, which is intricately linked to his idea of an “extended reality.” In terms of education, Freudenthal aims to ensure that students’ experiences help them internalize mathematical knowledge and view it as a seamless extension of their everyday life experiences. Based on this, it can be inferred that Freudenthal is actually more aligned with constructivism than it initially appears, despite his criticisms of it.

Freudenthal's perspective on mathematics education emphasizes the importance of viewing mathematics as more than just a series of steps or procedures, but as a dynamic human endeavor. While it is crucial to recognize that engaging in this activity also produces mathematical knowledge and concepts, this raises the question of how to design mathematics education that effectively combines these two aspects. To address this, Freudenthal proposed several concepts, including the concepts of "guided invention," "levels of learning processes," and "didactic phenomenology," all of which offer valuable insights into addressing this challenge.

The reinvention

According to the principle of reinvention, the learning process can be structured in a way that allows students to encounter and understand mathematics. Curriculum development begins with an idea or concept, and through experimentation and personal problem solving, students can arrive at their own solutions. The study of the history of mathematics can serve as a useful tool in this process, guiding students along the learning journey.

This approach, known as "guided reinvention," emphasizes the importance of the learning process itself rather than simply acquiring knowledge. It encourages students to take ownership of the knowledge they acquire and to feel responsible for it. To facilitate this, students should be given the opportunity to build their own mathematical knowledge bases based on their learning experiences. Freudenthal suggests that the history of mathematics can be a source of inspiration for students and that the principle of reinvention can also be influenced by informal solution methods. Often, students' informal strategies can be seen as anticipations or precursors of formal processes. This process of mathematization, similar to finding solutions, is a form of reinvention. When selecting contextual problems for students, it is important to choose those that allow for a variety of solution methods, preferably those that reflect a learning path.

Freudenthal believes that the inventive approach to teaching is an expansion of the Socratic method. He refers to "thought experiments" as a way of illustrating this, where textbook authors imagine themselves interacting with students and imagining their reactions and outcomes. These planned experiments involve anticipating students' reactions and devising strategies to address them. The goal is for students to reinvent the teaching topic through interaction and engagement. Freudenthal comments that while the student activity in the Socratic method is fictional, they should feel that their understanding and ideas are developing during the teaching process, with the teacher serving as a facilitator.

For Freudenthal, the Socratic method gives students a more active role in the process of constructing their own knowledge. However, there is a similarity between both approaches when it comes to anticipating and planning learning paths. This idea of anticipation and planning is discussed in relation to various challenges that need to be addressed, such as the mental activity of students and the necessary actions that need to occur for the process of reinvention to be feasible.

Freudenthal expands the idea of reinvention by introducing the concept of "progressive mathematization." This concept involves both the observer's perspective of reinvention and the student's perspective of experiencing "progressive mathematization" as an actor. Students begin by mathematizing a real-world topic and then move on to analyzing their own mathematical activity. This step is crucial as it includes a vertical component, explained by Freudenthal in relation to Van Hiele's theory, which states that activity at one level becomes the object of analysis at the next level.

The shift from "operator" to "object" means the transition from a procedure-based approach to a focus on the object itself, as observed by Sfard (1995) in the historical development of mathematics and the materialization described by Ernest (1991). Freudenthal's level theory forms the basis of realistic mathematics education (RME), which emphasizes the emergence of operational models in situational problem solving and their gradual transformation into entities that serve as models for formal mathematical reasoning (Gravemeijer & Terwel, 2000).

Phenomenology in didactics

Freudenthal emphasizes the importance of matching mathematical objects to the real-world phenomena they represent. In contrast to the concept acquisition approach, which involves using tangible materials to embody concepts, he suggests using phenomenologically rich situations: situations that are organized in a systematic way. In this approach, the selection of situations must be made carefully to ensure that they can be organized and understood using the mathematical objects that students are learning to construct.

The ultimate goal is to explore how the "object of thought" (nooumenon) describes and analyzes the "phenomenon" in a way that makes it accessible to calculation and thinking activities. This type of phenomenological analysis forms the basis of a didactic phenomenology that deepens the educational perspective of phenomenological analysis. For example, in order for students to understand the concept of length as a mathematical object, they must be confronted with situations in which length is an organizing principle.

Within the framework of phenomenological didactics, it is necessary to investigate the suitability of situations where a particular mathematical topic is applied, in order to determine its potential impact on the process of progressive mathematization. If we understand mathematics as a practical means of problem solving, it is reasonable to expect that current applications of mathematics involve problems that highlight these processes. Therefore, formal mathematics can be seen as a process of generalization and formalization of concepts and problem-solving procedures in various situations. The goal of phenomenological research, therefore, is to identify problematic situations that can be generalized and to discover situations that provoke paradigmatic solution procedures, which serve as a basis for vertical mathematization. By identifying phenomena that can be mathematized, we can better understand how they were originally conceived.

When considering research, Hans Freudenthal often asked himself what its purpose was, and he always concluded that the purpose was to bring about change. Education must continually adapt to the ever-changing society it serves. Therefore, the concept of "change" is more preferable than that of "reform," since what constitutes better education depends on the needs and priorities of society at a given time and how society evolves.

Education must change accordingly. In this sense, an important function of the researcher is to chart the path of change. Freudenthal believed that research should not be disconnected from the classroom, unlike traditional research. Rather, the search for the path of change should begin in the classroom. This philosophy of the goals and functions of research guided the approach to research at the Institute for the Development of Mathematics Education (IOWO), which Freudenthal directed.

At the time of the IOWO's creation, the predominant model in the German educational community was the R&D model. This model emphasized a separation between curriculum development and implementation, which contradicted Freudenthal's approach to "educational development." The concept of educational development, as Freudenthal saw it, encompassed not only curriculum development but also the ultimate goal of changing educational practice. Thus, educational development involved anticipating curriculum implementation from the outset as well as choosing a comprehensive approach to change that encompassed teacher training, counseling, developmental testing, and opinion formation, all based on the same educational philosophy.

In contrast to the curriculum movement, Freudenthal integrated research, development, implementation, and dissemination. As a result of this approach, he advocated the involvement of all stakeholders from the outset, under the banner of "educational development in dialogue with the field." The kind of change Freudenthal advocated was rooted in his belief that mathematics is a human activity. At the time IOWO was launched, however, little research had been done on this type of mathematics education. Therefore, questions about how to develop instruction had to be addressed during the development process itself.

Research for development

At first, our mathematician was reluctant to label IOWO's work as research. He believed they were observing as engineers, not as researchers. However, he later realized that this perspective separated research from educational development and failed to capture the interconnected nature of development in "developmental research." According to him, new knowledge must be justified by the process by which it was acquired.

The essence of developmental research lies in making the cyclical process of development and research consciously experienced and clearly reported. This enables others, such as teachers, to retrace the steps of the researcher in the learning process. Freudenthal emphasizes the importance of being constantly aware of the developmental process to ensure "traceability." In order for the results of developmental research to be credible and transferable, reflection on the developmental process must be informed.

The researcher should conduct thought experiments to understand how teaching and learning processes progress and then find evidence in teaching experiments to validate his or her expectations. Feedback from practical experience should drive an interaction between development and research. Ideas developed on paper should be put into practice immediately and classroom events should be consistently analysed and applied to further develop the work.

This process of deliberation and testing should result in a product that is both theoretically and empirically grounded. According to Freudenthal, developmental research can provide teachers with a framework to inform their own decisions. Within this framework, teachers can develop hypothetical learning trajectories that consider the current classroom situation as well as their own goals and values. Teachers can use this framework as a starting point, firmly rooted in the European teaching tradition, to guide their teaching.

Research conducted on national assessments has revealed that students in the Netherlands in the later years of primary school tend to achieve higher levels of success when interacting with modern texts compared to traditional ones. However, it is important to note that this trend does not apply to topics such as measurements and written algorithms. These findings suggest that the strategic approach of incorporating educational development in dialogue with the field, as implemented in the introduction of the Dutch curriculum and school textbooks, is the driving force behind this positive outcome. Indeed, retrospective studies examining innovations in mathematics education in both primary and secondary school have identified several key factors contributing to this success.

A key aspect of this plan would involve the review and renewal of the textbooks used in mathematics education. These textbooks would be carefully reviewed and updated to align with the new philosophy of mathematics education, incorporating innovative teaching approaches and engaging content. In addition, a thorough review of examinations would be undertaken to ensure that they accurately assess students' understanding and mastery of mathematical concepts. This would involve revising the format and content of examinations, as well as incorporating more open-ended and problem-solving questions that promote critical thinking and the application of mathematical knowledge. Finally, research and development would play a crucial role in driving innovation in mathematics education.

By continuously refining and improving the field of mathematics education, a dynamic and forward-thinking approach to teaching and learning would be ensured. To achieve significant and lasting improvements in mathematics education, a comprehensive and ambitious plan needs to be implemented. The plan would encompass several key components, including establishing a robust and transformative philosophy of mathematics education that empowers and inspires both students and teachers. It would also involve creating and refining a wide range of instructional sequences, examples, and prototypes that effectively engage students and facilitate their understanding of mathematical concepts.

These educational materials would be continuously developed and updated to reflect the latest advances in pedagogical techniques and educational research. And, the establishment of a mathematics education community would serve as a vital mediating infrastructure, facilitating the exchange of best practices, resources, and ideas among educators. The community would provide a platform for dialogue and collaboration, promoting the dissemination of innovative teaching strategies and approaches. To support the implementation of this plan, professionalization activities would be organized to enhance the skills and knowledge of mathematics educators, providing opportunities for ongoing professional development, as well as fostering collaboration and networking within the mathematics education community. To ensure widespread adoption and implementation of these improvements, efforts would be made to increase the accessibility and availability of high-quality mathematics education resources, providing comprehensive training and support to already in-service teachers, as well as developing and disseminating publications highlighting effective teaching methods and strategies.

Research development plays a crucial role in driving innovation strategies. Its main objective is to generate prototypes and theories that serve as valuable resources for teacher educators, textbook authors, and school consultants. These intermediaries, in turn, facilitate effective communication between researchers and teachers. The fundamental principle guiding educational development is the concept of engaging in meaningful dialogue with practical applications. Meaning, the Institute places great emphasis on involving diverse stakeholders, including teacher educators, consultants, textbook authors, researchers, test designers, and teachers themselves, in the research and development process from the very beginning. Rather than isolating itself in an ivory tower, the Institute recognizes the importance of incorporating real-world knowledge and experience into its innovative efforts.

It is not an easy task to locate Freudenthal's work in the contexts of didactics and curriculum studies because of his unique writing style, which lacks references to the authors who have influenced him. When it comes to didactics, Freudenthal often uses this term to describe the correct teaching and learning processes, which he believes should be rooted in reality. He firmly rejects the deductive approach, which he calls "the anti-didactic conversion."

According to Freudenthal, didactics is concerned with the processes involved in education. This aligns with Klafki's use of the same term, as draw inspiration from the phenomenological theory of Bildung as a pedagogical reform. Both start from the practice of education as a foundation and strive to overcome the exclusionary and elitist aspects of Bildung theory at certain points in their professional lives. Both emphasize the practical side of education and advocate comprehensive schooling as a necessary reform. However, Klafki focuses primarily on planned lessons and lesson preparation, where the learning process may not be entirely real. Klafki's fundamental questions revolve around the content of Bildung, while he pays less attention to teaching methods and processes.

Freudenthal mentions the term "curriculum," although he does not use it as frequently as the word "didactics." In regard to his perspective on curriculum development and the role of theory, there is a remarkable resemblance to the work of Joseph Schwab, who occupies an important position in American curriculum theory. In a similar vein, but without any influence from Schwab, Freudenthal stresses the unique nature of curriculum work and the importance of dialogue between curriculum experts and teachers.

He strongly opposes the idea of a rigid curriculum system and firmly rejects the concept of packaging and organizing content into predetermined structures. This view is particularly noteworthy at a time when curriculum theory was influenced by a behavioural approach, and the R&D (Research, Development and Diffusion) method was hailed as the ultimate solution in Germany and the Netherlands.

Freudenthal also advocates that mathematics be viewed as a human activity and encourages guided reinvention. This humanistic, practical, process-oriented, phenomenological, and pedagogical reform philosophy, which is widely discussed in the context of curriculum development, distinguishes Freudenthal's stance from that of many of his contemporaries in the field of mathematics education. His beliefs often clashed with those of behaviorist-oriented psychologists such as Bloom and proponents of the "new mathematics" movement, who proposed the development of a mathematics curriculum based on an abstract deductive system.

Freudenthal, who was educated in and influenced by the German Bildung tradition, rejects the idea of an exclusive form of education reserved for an elite group separated from the masses. Instead, he strongly advocates "mathematics for all" and strives to make mathematics accessible to all individuals. He condemns any form of conforming to societal norms and aligning oneself with the inevitable effects of mathematical concepts.

He strongly believes that students with different ability levels in the early years of secondary education, which is typically between 12 and 15 years old in the Dutch context, should not only be in the same class but also follow the same curriculum. Consistent with his pedagogical beliefs, he emphasizes the importance of forming diverse learning groups. Many of Freudenthal's ideas remain the subject of ongoing debate.

Psychologists, who view learning as an informational process, strongly oppose educational theories of this nature. Similarly, there are instances of opposition within the mathematics education community to the fundamental idea that students must make the transition from the real world to the world of mathematics. Critics argue that drawing from real-life experiences and reinventing mathematical concepts is a waste of time. However, it is important to note that those who oppose Freudenthal's ideas have limited empirical evidence to support their view. Several teaching experiences have demonstrated the value of the Realistic Mathematics Education (RME) approach. Furthermore, numerous studies investigating the effects of mathematics curriculum influenced by Freudenthal's ideas have shown that learning mathematics in real-life contexts and within diverse groups is feasible and effective.

The impact of Freudenthal's ideas is evident in all the Dutch texts. Moreover, there is practical and empirical evidence supporting the feasibility and efficiency of the EMR approach. One of Freudenthal's most convincing arguments in favour of EMR is that not all students will become mathematicians in the future; instead, they will primarily need mathematical skills that help them solve problems in everyday life situations.

Chapter 2

Realistic mathematics education

Mathematics is interconnected with the world and the image of a mathematician is shaped by societal perceptions. The perspective is now reflected in many countries' curricula and in the Programme for International Student Assessment (PISA) assessments. Mathematical literacy, according to PISA, is an individual's ability to recognise and understand the role of mathematics in the world, make informed mathematical judgements and use mathematics in a way that meets their present and future needs as a responsible and reflective citizen.

This view emphasizes the importance of mathematics in society and its practical application in diverse contexts. Freudenthal's influence extends beyond his academic career. He played a major role in the International Group on Psychology and Mathematics Education, the journal Educational Studies in Mathematics, and the International Commission for the Study and Improvement of Mathematics Teaching. Through his numerous writings, he expressed his opposition to the pedagogical and teaching approaches that emerged in the mid-20th century, including Bloom's taxonomy, structured assessments, quantitative methods in educational research, the direct application of Piaget's ideas in the classroom, the separation of educational research, curriculum development, and teaching practice, and the introduction of modern or established mathematics into schools.

Realistic Mathematics Education (RME) is an educational approach developed since the late 1960s by Hans Freudenthal and his colleagues at the Freudenthal Institute for Mathematics and Science Education at Utrecht University in the Netherlands. Hans

Freudenthal, a leading mathematician specializing in topology, algebra, and the history of mathematics, was forced to emigrate from Germany due to the rise of the Nazis. In the Netherlands he set about promoting changes in mathematics education not only within the country but also in other European nations.

Freudenthal has been publishing on mathematics education since 1948. Over the years he worked with the Institute for the Development of Mathematics Education (IOWO), which he founded in 1970 at Utrecht University, together with other collaborators. This institute has laid the foundation for the current development and expansion of the Educational Materials and Resources (EMR) programme. Treffers (1987) describes the principles on which EMR is based:

- Principles include a focus on phenomenological exploration, where students are exposed to rich and meaningful phenomena to develop an intuitive understanding of mathematical concepts.
- The use of models and symbols is also emphasized, as students move from informal, context-bound notions to more formal mathematical ideas.
- Students' own constructions and productions are valued and used in the teaching process, since their personal experiences contribute to meaningful learning.
- Interaction is another key aspect, as students can compare and contrast their contributions, reflecting on the process of mathematization.
- It is also important to link curricular themes and axes since the connections between different areas of mathematics are considered when teaching specific topics. For example, when teaching statistics, the necessary algebraic or scientific knowledge is considered, and when introducing the notion of distribution, it is linked to other statistical concepts.

During a presentation addressed to educators in the field of mathematics, Freudenthal states that mathematics involves problem solving and the organization of objects of study, which may be real-world phenomena that require the organization of mathematical patterns to solve problems. Alternatively, they may be mathematical issues, whether new or old, one's own or someone else's, that need to be organized with new ideas to achieve better understanding in a broader context or through an axiomatic approach. He goes on to discuss how children are initially taught mathematics as an activity, but as they mature, they are often presented with a pre-constructed, well-organized mathematical system under the assumption that rational individuals will understand deductive systems. However, this approach is not effective.

For Freudenthal, transmitting ready-made mathematics, which is the product of mathematicians or textbook authors, is counterproductive in terms of teaching. Instead, he suggests teaching students to mathematize. Treffers (1987) further expands this concept by differentiating between two dimensions of mathematization: horizontal and vertical. Horizontal mathematization involves transforming a real-world problem into a mathematical problem using common sense, intuition, observation, empirical approximation, and inductive experimentation.

On the other hand, vertical mathematization involves navigating within the realm of mathematical reality through schematization, generalization, proof, rigor, and symbolization. Horizontal mathematization leads from the world of life to the world of symbols, where individuals live, act, and experience, while vertical mathematization involves the creation, recreation, and manipulation of symbols in a mechanical, comprehensive, and reflexive manner (Zolkower & Bressan, 2012). It is important to note that the boundaries between these two worlds are not clearly defined and can expand or contract depending on several factors.

To teach students how to apply mathematical concepts to real-life situations, it is important to engage them in guided activities that involve organizing realistic problems. The terms "realistic" and "reality" are used in this context to refer to situations that align with common sense and are perceived as genuine within a given scenario.

In the early grades, we focus on familiar everyday contexts and situations involving numbers, such as people getting on and off a bus. As students become more familiar with numbers and their relationships, their understanding of what is real or meaningful to them expands. It is important to note that the term "realistic" is often misunderstood in a narrow sense, which is due to the choice of this name. In Dutch, "zich realis-eren" means to imagine. Therefore, in a broader sense, a situation is considered realistic as long as it is presented to the individual as feasible, reasonable, or imaginable. For example, when we teach geometry and measurement, estimation, ratios, and proportions, we can draw inspiration from works of fiction such as "Gulliver's Travels."

The goal of mathematics education, according to Freudenthal, is to develop in students a mathematical disposition that includes a variety of skills and abilities. This includes the ability to identify the essential aspects of a situation, problem, procedure, algorithm, symbolization, or axiomatic system. It also involves recognizing common features, analogies, and isomorphisms, as well as providing examples of general ideas and discovering new objects and operations.

Students should be encouraged to find shortcuts, develop new strategies, invent new symbolizations, and reflect on their own thinking by considering different perspectives or points of view. In addition, mathematical readiness includes using functional language and conventional variables, determining the appropriate level of precision for a given problem, identifying mathematical structures in a context, and recognizing when it is not relevant or appropriate to use mathematics. Therefore, students should consider their own activity as an object of reflection in order to advance their understanding to a higher level.

To develop this mindset it is necessary to go through a teaching-learning process that involves a guided reinvention, as described by Freudenthal (1991). The goal of this process is not simply to teach mathematics, but rather to teach students how to think mathematically, how to abstract concepts, how to create schemes, how to formalize formulas, how to algorithmize procedures, and how to express mathematical ideas in verbal form.

This approach to teaching, known as guided reinvention, is based on the principles of instructional phenomenology, which involves looking for real-life contexts and problem situations that foster mathematical thinking. By examining the ways mathematical objects are used and understood in everyday language and situations, educators can develop localized theories for teaching these concepts.

Didactic phenomenology is based both on the History of Mathematics, considering the crucial moments in the development of mathematical ideas and their evolution over time, and on the unique thoughts and creations of the students themselves. Thus, the EMR (Teaching and Mathematization of Reality) approach considers learning as a non-linear process involving progressively higher levels of organization, abstraction, generalization and formalization.

The transition from one level of learning to another, which usually occurs suddenly and means a break in learning, involves the use of a model to symbolize a situation. Gradually, this model becomes detached from the original situation and becomes a tool for organizing similar situations. There are four levels involved in this distinction between a model of and a model for: situational, referential, generalization, and formal.

- At the situational level, strategies develop spontaneously to organize the problematic situation.
- The reference level introduces graphic models, notations and procedures that represent the problem but are still connected to the specific situation.
- The general level is reached through exploration, reflection and generalization, which moves away from any reference to the context.
- Finally, the formal level involves working with general and conventional procedures and notations that are disconnected from their original contexts.

To foster these processes, it is important to work on problems that can be solved using different tools and to encourage the use of multiple strategies and procedures. Thus, students' work on these problems can reveal their understanding and arithmetic skills at a particular moment in time, which is valuable for making instructional decisions. This information not only helps make small-scale decisions but also guides larger-scale decisions. The class's collective understanding and problem-solving strategies provide a snapshot of its learning trajectory. The strategies used by individual students offer insight into the long journey the class will take. What is happening in the classroom at any given moment provides a glimpse of what is to come and what is to come.

The Contexts

Context refers to a specific aspect of reality that is mathematized during a learning process; they are not artificial disguises for mathematical content, but rather real-life situations that curriculum designers and teachers present to students to encourage them to apply mathematical concepts. Freudenthal argues that viewing context as a distraction from the mathematical message is a mistake, since context itself is the message and mathematics is the tool used to understand it.

When a context is meaningful to a student, it serves as a starting point for his or her mathematical activity, drawing on his or her common sense and suggesting the use of informal strategies relevant to the situation. It is important to note that the realism of a context depends on the student's prior experience and ability to imagine or visualize it. For example, a first-grade student may find it as "real" to work with situations involving changes in the number of passengers on a bus during different routes as he or she would later find it to work with arrows as symbols representing such changes in later years.

Such contexts pave the way for higher-level mathematical concepts such as operators and equations. Streefland (1991) supports this idea by describing a research project on teaching fractions that begins with the concept of fraction and ratio simultaneously by mathematizing situations involving equal distribution, such as distributing 5 chocolate bars among 6 children.

Realistic contexts serve two functions: first, as a resource for generating mathematical ideas and second, as a domain for applying mathematical concepts. By using meaningful real-life situations as a starting point, students can bridge the gap between reality and mathematics through interactions with peers, teacher guidance, and the use of appropriate models that emerge from their own thinking. This approach allows students to develop skills such as structuring, organizing, symbolizing, visualizing, and schematizing. They can also progress in their mathematical understanding by improving the efficiency of procedures, using shortcuts, and making the transition from colloquial language to the conventional language of symbols and variables.

From this perspective, it is also argued that in order to improve students' mathematical thinking skills and, consequently, to improve the general mathematical competence of individuals, it is imperative to critically analyze and explore the connections between mathematics and its applications (both positive and negative) in various areas such as science (including social, natural and exact sciences) and technology.

The Models

Numerous models have been developed within the work carried out at the EMR. These include models such as money, the rekenrek (a two-coloured abacus with 20 balls arranged in two identical rows) and paradigmatic situations such as the collective, which is represented by arrows to symbolise dynamic situations before and after. Other models include the "pancake house", the parents' meeting at school and the candy factory with 10-packs.

Models such as the circular model, the double or percentage bar, and the ratio table have also been explored. Two-coloured necklaces structured in groups of 10 have also been used, which led to the development of the "open" number line as an arithmetic model. In addition, the number line has been used as a model for solving linear equations, and the notebook notation and the combination table have been used to work with systems of two equations with two unknowns.

The use of these models, among others, is essential to counter one of the greatest challenges in mathematics teaching, which is the tendency towards algorithmization and premature formalization. Thus, models play a crucial role in simplifying complex realities or theories, allowing for mathematical treatment. They emerge and develop through a guided reinvention process and can be applied to diverse contexts.

EMR has worked on numerous models, including money, rekenrek, paradigmatic situations, and various arithmetic and equation-solving models. The use of these models is important to combat the negative effects of algorithmization and premature formalization in mathematics teaching. According to Freudenthal, a model serves as a means of simplifying and idealizing a complex reality or theory, making it more amenable to mathematical treatment. It is not a pre-existing artifact or representation, but rather an entity that emerges and evolves through a process of guided reinvention. Initially, models are closely tied to the specific contexts and situations in which they arise, but over time they become decoupled and take on characteristics of formal, general models that can be applied to a variety of contexts, both within and outside of mathematics. This transition involves moving from being a "model of" a particular situation to being a "model of" mathematical reasoning in a variety of situations.

Models advocate for an increase in the use of mathematical concepts in a way that is relatable and understandable to students; they should be flexible enough to be applied in more advanced or broader contexts, while still allowing students to understand their initial meaning and purpose. It is important that models support both vertical progression in mathematical understanding and the ability to connect to the original context or situation. This allows students to fully understand the meaning and significance of their actions within the model. Models should behave in a natural and obvious way, aligning with informal strategies and being applicable to a wide range of scenarios. As an example, the percentage bar initially emerges from a specific context such as parking lots or movie theaters, where it represents the occupation of spaces through shading. Over time, the percentage bar is detached from its original context and transformed into a formal tool that can be used to work with and reflect on percentages.

The ratio table, the bar model, and the double number line are schematic models that differ from traditional algorithms because they keep important aspects of context visible. These models allow for recording intermediate steps and are easily adapted to each student's level. They also suggest the use of shortcuts and multiple strategies for problem solving. By using these three models simultaneously, we can examine the advantages of each for different types of problems and explore mathematical relationships within them. In addition, the combined table and notebook notation are notable tools for algebraization proposed by EMR.

These tools give students the ability to understand traditional methods, such as understanding the components of a system of equations, identifying what they are looking for, recognizing equivalent equations, understanding why certain systems may have one, multiple, or no solutions, and determining the most appropriate method for finding these solutions. In general, the transition to working with pure systems is not a challenge for these students, and in case they encounter difficulties, they can use these models to recall the typical situations that led to their creation. This allows them to redefine the operations they perform at a formal algebraic level.

The interaction

In RME, reflecting and mathematizing are closely related. According to Freudenthal, students need to be able to reflect on their own activity in order to reach the highest level of understanding. In guided reinvention processes, interaction between teacher and students is crucial to promote reflection and the exchange of ideas. The classroom should provide a space for individual, group, and collective action and reflection, where students not only answer questions and solve problems, but also formulate their own mathematical questions, share and evaluate ideas and solution methods, and symbolize and generalize mathematical relationships.

Typically, sharing in a mathematics class occurs after the problem has been solved, but under the guidance of a trained teacher, sharing can take the form of “thinking together out loud” in the present tense and subjunctive and conditional moods, allowing for the sharing of ideas in the process of development. The question arises as to how these types of conversations can help students fully understand and engage in the task of mathematizing.

In the field of EMR, there is a significant emphasis on problem formulation and solution. However, the focus is not solely on teaching students how to solve specific problems, but rather on cultivating their ability and inclination to apply mathematical concepts and methods in a variety of contexts (such as arithmetic, geometry, algebra, and formalization). To achieve this goal, teachers should present open-ended questions that are within the reach or imaginable of their students, and they should strive to understand students' thoughts and reasoning processes.

The teacher must value and take a genuine interest in student input, encouraging interactive situations in both whole-class and small-group settings. It is crucial that the teacher builds on student ideas, guiding them through reflective processes that promote higher levels of mathematical thinking and understanding for each individual student and for the class as a whole. This requires a teacher who can anticipate key developmental milestones along the path of progressive mathematization.

If the main activity of students is to engage in mathematics, then what is the main activity of teachers and professors? According to Freudenthal, their main activity is to organize and structure the teaching process, which has both a horizontal and a vertical aspect. The process of teaching mathematics is parallel to the process of mathematizing. It involves becoming aware of the didactic reality and creating a framework for teaching, on the one hand, and developing a deeper understanding and generalizing from teaching situations, on the other. Horizontally, teachers focus on the teaching and learning phenomena that occur in their classrooms and in other classrooms. Vertically, they reflect on these situations and use them to enhance their own teaching strategies and techniques to support the mathematization process.

The theoretical bases: EMR

Realist Mathematics Education, as an international movement, was founded by Hans Freudenthal, a German mathematician and educator. The movement emerged in the 1960s as a response to the mechanistic approach to teaching arithmetic and the use of "modern" or "conjunctitarian" mathematics in classrooms. Today, many of Freudenthal's original ideas are adopted and discussed in current educational theories and have influenced the curricula of a number of countries, including the United States, Japan, Indonesia, Great Britain, Germany, Denmark, Spain, Portugal, South Africa, Brazil, and Puerto Rico.

A fundamental principle of EMR is that mathematics education must be grounded in reality, relevant to students, and meaningful to society in order for it to be valuable to human development. According to Freudenthal, the perception of mathematics is intertwined with our perception of the world, the role of mathematicians is linked to our understanding of humanity, and the teaching of mathematics is connected to society as a whole. In his view, a critical consideration during his time was whether mathematics should be seen as a subject for a select minority or as a subject for all individuals. He believed that it is crucial for all students to have some level of engagement with mathematical work, which he defined as the act of organizing reality using mathematical concepts and tools, including mathematics itself.

Mathematizing is a step-by-step process that involves various actions such as:

- Identify important features in different situations, problems, algorithms, formulas, symbols and systems based on axioms.
- Find common points, similarities, analogies and isomorphisms between these elements.
- It also involves providing concrete examples to illustrate general ideas and concepts.
- It requires approaching difficult situations in a systematic and exemplary manner. In addition, mathematization involves the sudden appearance of new objects and mental operations that help in problem solving.
- It involves searching for efficient strategies and finding ways to simplify initial approaches and symbolizations to create formal diagrams, algorithms, symbols and systems.
- Finally, mathematizing involves reflecting on the entire mathematical process, considering the various phenomena involved from multiple perspectives.

Practical scenarios and challenging circumstances

A context refers to a specific domain of reality that is revealed to students during the learning process in order to be mathematized. Mathematics evolved as a means of mathematizing real-life situations in the natural and social environment, and therefore its teaching should also be based on the organization of such situations. However, this does not imply focusing solely on perceptual phenomena, as this would restrict students' opportunities to gain experience and engage with mathematics itself.

The goal is for students, who may initially lack sufficient mathematical skills, to reinvent these tools by tackling problems presented in realistic contexts and situations. A context can take the form of an event, proposition, or situation derived from reality that is meaningful to students or can be imagined, prompting the use of mathematical methods based on their own experiences. It provides concrete meaning and support for relationships and operations that are relevant to mathematics.

These situations can be drawn from everyday experiences, such as bus routes or shopping and money management. In addition to contexts derived from daily life, mathematics itself offers contexts within the realm of problems involving pure numbers and numerical relations, such as the context of prime numbers. There are several types of contexts, including real, artificial (fantasy), mathematical, and virtual, each of which originates in reality but incorporates non-real elements for purposes of simplification or simulation.

In the field of mathematics education, it is crucial to recognize the important role that realistic contexts play in the student learning process:

- Realistic contexts serve as a foundation for teaching and learning, allowing students to develop mathematical concepts and apply them across a variety of domains.
- When carefully selected, these contexts capture students' interest, fostering engagement and motivation.
- Realistic contexts serve as tangible objects of study, facilitating the accessibility of mathematical content for students at different levels of understanding.

- By incorporating real-world scenarios, students are encouraged to use their common sense and leverage their informal knowledge to build mathematical models.
- The openness of these contexts, allowing for multiple strategies and solutions, fosters meaningful mathematical discussions among students.
- These realistic contexts are explored in a comprehensive and in-depth manner, ensuring a deep understanding of the mathematical concepts involved.

While it is important to consider the relative nature of the realistic context to avoid over-generalizations and over-simplifications, the realism of a context depends on students' prior experiences and their ability to imagine or visualize it; it is beneficial to use the models that emerge from students' own mathematical activities as tools to represent and organize these contexts and situations.

These models serve as intermediaries through which complex realities or theories are idealized or simplified for formal mathematical treatment. It is crucial to note that in the context of EMR, the term "model" does not refer to pre-existing models imposed from formal mathematics, but to emergent models that develop during the teaching-learning process. And they are formed through the organization and reorganization of activities that arise from problematic situations. Initially, these models are intricately linked to the specific contexts and situations from which they arise, but over time they become decoupled and take on characteristics of formal and general models. As a result, they can be applied to diverse contexts and situations, moving from being a "model of" a particular situation to a "model for" mathematical reasoning in both mathematical and non-mathematical settings.

Models in the field of EMR serve not only as representations, but also as tools for analysis and reflection. They are used to perform various actions and operations, as well as to visualize, explain, compare, contrast, and verify relationships. To serve these purposes, these models must meet a number of crucial criteria:

- First, they must be based on realistic and imaginable contexts.
- They must have enough flexibility to be applicable to more advanced or general levels.
- Unlike traditional teaching methods, where models are fixed, these models are subject to change over time. This dynamic nature allows for progression in the mathematization process, while also allowing students the ability to revisit the original situations from which the strategies were derived. This ability to move between levels is what makes these models particularly powerful.
- Finally, these models must be viable, that is, they must behave in a natural and obvious way. They must align with students' informal strategies, as if the students themselves could have discovered them independently, and they must also be easily adaptable to different situations.

It is important to note that informal solutions and independent creations by students play a central role in the teaching and learning process. By working on problems that can be solved in multiple ways, students' levels of understanding and computational skills at a given time can be revealed. This information is crucial not only for making small-scale teaching decisions but also for guiding larger-scale educational decisions.

A snapshot of the classroom, with its various levels of understanding, provides insight into the trajectory of learning and teaching. The solution strategies employed by individual students collectively expose essential elements of the long-term path that students will undertake. Thus, what is observed in the classroom today anticipates what is to come and beyond. Instructional phenomenology involves initially studying the different ways in which a mathematical concept, such as fractions, ratios, functions, proportions, and angles, manifests itself in real life. This includes considering how these concepts are commonly referred to in everyday language. From this understanding, the didactics of the topic can be built.

The EMR has experimented with several classroom models that are easily presented through contextual situations and can be recreated by students. These models include manipulative teaching materials such as tokens, money, and necklaces with two-colored balls arranged in groups of ten. In addition, paradigmatic situations such as the bus, the pancake restaurant, the parent meeting, the 10-unit candy factory, and the location of a fire have been used.

Diagrams such as the circle model, the double bar or percentage model, and the ratio table have also been used, as well as diagrams such as the tree and path diagrams. Notational forms such as arrow language, notebook notation, and the combination table have also been used to solve systems of equations with two unknowns, and symbolically expressed procedures such as algorithms or column formulas. The exploration of contexts and models that naturally lead to the use of mathematics is known as didactic phenomenology, a concept coined by Freudenthal. This approach is strongly influenced by the history of mathematics and the ideas and creations of students that arise during the teaching process.

The role of the teacher

In the context of EMR, mathematics instruction should involve a guided reinvention approach. Students should have opportunities to independently discover mathematical concepts and skills by organizing and structuring real-life problems. During this process, students interact with their peers and receive guidance from the teacher. Effective learning in this approach involves explicit negotiation, intervention, discussion, cooperation, and assessment.

Informal methods serve as a foundation for students to eventually understand formal mathematical concepts. This interactive teaching method requires students to explain, justify, agree or disagree, question alternatives, and reflect on their thinking. The teacher plays a crucial role as a mediator, facilitating communication between students and the problems they encounter, as well as facilitating communication between students themselves. Furthermore, the teacher bridges the gap between students' informal problem-solving approaches and the established formal tools of mathematics.

The act of learning mathematics is considered a social process in which individuals come together to reflect collectively, resulting in deeper understanding. Both vertical interactions between teachers and students, and horizontal interactions between students, play a crucial role in this process. How the teacher manages these interactions is key to maximizing opportunities for students to generate, exchange, and understand ideas.

It is important to note that a class is not considered a homogeneous entity, but rather a group of individuals who follow their own unique learning paths. However, this does not mean that the class is divided into groups with similar processes. Instead, the class remains together as an organizational unit or engages in cooperative work in diverse groups, as Freudenthal advocates.

By selecting problems that suit different levels of understanding, all students can work on them. Additionally, there is a strong emphasis on integrating the various strands or units of the mathematics curriculum. Solving real-life problems often requires making connections and using a wide range of mathematical concepts and tools.

The EMR curriculum avoids strict distinctions between curricular strands, creating a more cohesive approach to teaching and allowing for different methods of mathematizing situations using various models and languages. This ensures a high level of coherence across the curriculum, rather than teaching each strand in isolation and ignoring the connections between them. In practical applications, problem solving typically requires more than just knowledge of arithmetic, algebra, or geometry.

Phenomenology

Freudenthal's perspective on mathematical objects differs from traditional mathematical philosophies such as realism and Platonism. These philosophies believe that mathematical concepts exist independently of human activity and are discovered through mathematical exploration. Freudenthal, however, maintains that mathematical concepts are created and constructed through mathematical practice. He suggests that mathematical objects are not just tools for organization, but real objects with their own properties and actions.

As these mathematical objects are brought into the world, the world itself expands and grows. Mathematical concepts and ideas are used to organize phenomena of both the real world and mathematics. On the other hand, mental objects are creations of individuals from their experiences and serve as a means to organize and understand their own experiences. Freudenthal also recognizes the challenge of teaching mathematical concepts as they require instilling the corresponding concepts in the minds of students. This analysis aims to explore the scope and method proposed by Freudenthal in his theory of Didactic Phenomenology of Mathematical Structures for teaching and learning mathematical concepts. It involves examining the relevant literature and understanding the key ideas put forward by Freudenthal.

The term phenomenology, as used here, does not refer to the interpretations given by philosophers such as Husserl, Hegel, or Heidegger. Rather, it belongs to the Greek origins of the word, where "phainomeno" means "that which appears." In this context, phenomena are appearances or how things appear to us. In the realist philosophical tradition, the world of noumena is considered the real world.

The contrast between phenomenon and noumenon represents a contrast between two worlds: the world of appearance and experience (phenomenon) and the world of the sensible and intelligible (noumenon). Some philosophers hold that mathematical concepts are noumena, which places them outside the realm of our experience. However, this contradicts the ideas of Freudenthal, who sees mathematical concepts as a means of organizing phenomena. According to this view, mathematical concepts are part of the field of phenomena that are organized by new mathematical ideas.

Mathematical concepts are therefore not separate from our experiences or in a separate world from the phenomena they organize. They are in fact objects of our mathematical experience. To engage in phenomenology, one must describe the relationship between these series or pairs: the phenomenon and the means of organization. The process of creating mathematical objects involves the means of organization becoming objects that appear in the field of phenomena.

Thus, mathematical objects become incorporated into our experiences and become part of a new relationship between phenomenon and means of organization. This iterative process continues and leads to the creation of new mathematical concepts and the generation of increasingly abstract mathematical objects. The phenomenology of a mathematical concept, structure, or idea involves describing the noumenon (the concept itself) in relation to the phenomena it organizes. This includes identifying the phenomena for whose organization and extension the concept was created, understanding how it acts as a means of organization for these phenomena, and recognizing the power it gives us over these phenomena.

Concepts, known as noumena, are intricately connected to the phainomenon. When we look at the didactic element in this relationship, specifically how the concept R phenomena are acquired in the teaching and learning process, we enter the realm of didactic phenomenology. This field explores the phenomena that exist in the world of students and those that occur in teaching sequences, particularly in the context of mathematics.

By examining the fRc relationship in terms of students' cognitive growth, we engage in genetic phenomenology. Here, the focus is on how phenomena are perceived and understood in relation to students' cognitive development. Furthermore, if we explore the historical acquisition of this fRc relationship, we enter the realm of historical phenomenology. In this case, we investigate the phenomena for which the concept was originally created and how it was subsequently expanded to encompass other phenomena.

The suggested order for studying these phenomenologies begins with pure phenomenology, gaining knowledge of mathematics and its practical applications. This is followed by historical phenomenology, which provides information on the formation of these relationships throughout history. Next, we delve into didactic phenomenology, understanding the teaching and learning process, and finally, genetic phenomenology examines the cognitive growth of students. It is important to note that the description of the relationships between the phenomenon and the concept considers both the relationships established in the first case and how these relationships were developed, acquired or formed in the educational system, cognitively or historically in the other three cases.

Phenomenology, which can be defined as a method of analyzing mathematical content, involves the phenomenological analysis of mathematical concepts or objects. This analysis is carried out with a didactic intention, that is, it is carried out prior to any curricular design or development and is considered a component of didactic analysis. The purpose of phenomenological analysis is to serve as a basis for organizing the teaching of mathematics, rather than attempting to provide an explanation of the nature of mathematics.

One of the main tasks of phenomenology is to investigate phenomena that organize mathematical concepts by analyzing them. These phenomena are assumed not to have existed before. In contrast to the typical approach to teaching mathematics, Didactic Phenomenology proposes a different approach. It suggests starting with phenomena that require organization by a concept and then teaching students how to manipulate these means of organization.

Instructional phenomenology should be used to develop plans with this type of approach. For example, when teaching about Groups, instead of starting with the concept of Group and trying to materialize it, the focus would be on examining the phenomena that could lead the student to form the mental object that is being mathematized by the concept of group. If the necessary phenomena are not available at a certain age, attempts to inspire the concept are abandoned.

In the school system, concepts are often introduced to students before they have any experience with the phenomena involved. The educational system aims to help students form mental objects as a means of organizing these phenomena, and it also provides access to the means of organization that history has provided, which are concepts.

In the context of history, mathematical concepts do not exist prior to our experience with them. It is the activity of mathematicians that creates these concepts. Throughout history, mathematical concepts have emerged as consolidations of mental objects. Mathematical activity generates concepts from mental objects. The relationship between concepts and mental objects is complex, as both serve as means of organizing phenomena. Mental objects preexist concepts, and concepts do not replace mental objects, but rather allow the formation of new mental objects that contain or are compatible with them. The distance between the initial mental object and the concept can be significant.

Chapter 3

The contextualization of realistic mathematics in education

The demand to establish a connection between mathematics taught in educational institutions and the lives of students is a call from society, coming from both the academic and professional worlds. This demand is not isolated; it is part of a broader request to the school system itself, where society as a whole expects that what is taught in our schools will enable students to function effectively in their lives. There have been swift responses to these demands.

At the international level, the Organization for Economic Co-operation and Development (OECD) has highlighted, through the PISA study, the importance of developing mathematical skills that enable individuals to recognize and understand the role of mathematics in the world, to reason well-foundedly and to use mathematics according to their vital needs as constructive, engaged and reflective citizens. More recently, the same study emphasises that the development of a mathematical culture in schools should help individuals to identify and understand the role of mathematics in the world, providing them with the necessary judgement to make decisions based on becoming constructive, engaged and reflective citizens.

The National Council of Teachers of Mathematics (NCTM) also emphasizes the importance of connecting the mathematics taught with the current and future lives of students. In the Latin American context, this social demand is also evident. The curricular guidelines established by the Colombian Ministry of Education state that the main objective of mathematics education is to help people make sense of the world around them and understand the meanings constructed by others.

By learning mathematics, students not only develop their capacity for logical thinking and reflection, but they also acquire powerful tools to explore, represent, explain and predict reality, allowing them to act in and for it (Ministry of National Education, 1998). Venezuela's proposal for teaching all subjects in the curriculum also emphasizes the importance of linking education to the lives of students, making it relevant both individually and socially. In addition to these demands from society, there is also a demand from the academic community. Within the field of mathematics education, the need to establish this connection is defended from different epistemic perspectives, such as Critical Mathematics Education and Realistic Mathematics Education. These demands are justified by the desire for mathematics education to contribute to the formation of conscious and participatory citizens, promoting inclusive mathematics and avoiding the exclusive nature of traditional approaches.

When focusing on educational institutions, it is increasingly common for people, particularly teachers, to emphasise that “mathematics is everywhere”. This is a response to the growing societal demand for school mathematics to be relevant and applicable to students’ present and future lives. However, while there is widespread agreement on the importance of connecting mathematics to students’ lives, it has proven difficult to meet this expectation. It is common and well-established that when both pre-service and in-service teachers are asked to provide specific examples, they often struggle to do so. Their answers often revolve around basic concepts related to buying and selling, but there are many other phenomena and life situations in which mathematics plays a role.

In other words, mathematics must be meaningful to the students who learn it by relating it to their needs and interests in their own life experiences. To achieve this, teachers must create opportunities for ongoing communication with their students, and the discourse they employ plays a crucial role in this process. When we refer to teacher discourse, we are considering it in a pragmatic sense, considering the contextual relationships that guide the communicative interaction between the teacher and his or her students. What we are advocating is the need to endow mathematics education with meaning that resonates with the students who learn it.

To ensure that mathematics is meaningful to students, it is essential to connect it to their everyday lives. This connection must be both personal and social. Therefore, we propose to prioritize the contextualization of mathematics teaching. This idea is not new, as several researchers and educators have emphasized the need for this approach. They argue that mathematics should be seen as a human activity and should be taught in relation to students' reality. Instead of seeing mathematics as a deductive system, these scholars suggest that students should engage with mathematics through real-life experiences that help them see it as a tool to organize and understand their present and future realities.

This perspective remains relevant in current debates about curriculum design, as well as in the views of organizations such as the NCTM and the OECD. From an individual point of view, each person learns best when knowledge is meaningful to his or her own life. From a societal perspective, mathematics education should have practical applications that enable individuals to integrate into society.

The information collected provides evidence of educational practices in mathematics that incorporate real-life situations into the teaching processes. However, these practices can be seen as deviations that should be avoided in our classrooms. We will discuss three specific deviations: teachers' understanding of reality, the types of relationships that are established between mathematical concepts and real-life situations, and the depth of the study of mathematical objects in the classroom.

It is important to reconsider certain educational practices in mathematics that incorporate real-life situations. Teachers need to have a broader understanding of reality, not just limiting themselves to everyday situations, and they should prioritize the integration of mathematics with students' life experiences from the beginning. By adopting a more comprehensive approach that combines theory and practice, mathematics can become more meaningful and relevant to students, regardless of their socioeconomic background.

The first deviation concerns teachers' understanding of reality and how it influences the integration of mathematics with real-life situations. In interviews with mathematics teachers, both pre-service and practicum, they were asked to provide examples of real-life situations that could be linked to mathematics. Without exception, these teachers gave examples that were perceptible to the senses and closely related to the students' everyday lives. This finding is consistent with other studies that have also observed this tendency among mathematics teachers. However, limiting the context to only everyday situations have ethical and pedagogical implications. It primarily disadvantages students who have limited life experiences due to their socioeconomic disadvantage.

The context can be geographically and chronologically close to the student, but it should not be limited exclusively to his or her immediate environment. For example, studying Mayan culture and its contributions to mathematics may not be an everyday context for students, but it can still be meaningful and interesting to them if they are properly motivated. Exploring mathematics in different cultural contexts can broaden students' horizons, regardless of their socioeconomic background.

The second deviation relates to the types of relationships that are established between mathematical concepts and real-life situations. Many teachers tend to present mathematical theory first and then try to show its application in specific contexts. This deductive approach to learning assumes that students can only understand and learn theory if it is disconnected from their life experiences. However, presenting mathematics in a theoretical way without relating it to students' real-life experiences can make it less interesting for them and difficult to understand the subject. We believe in a teaching approach that combines theory and practice, where mathematics is integrated into students' lives based on their previous experiences. This approach enables students to see the relevance and applicability of mathematics in their daily lives and broadens their worldview.

Furthermore, we want to highlight another aspect that deviates from our expectations. This concerns the degree to which mathematical concepts are explored in the classroom, specifically through the incorporation of real-life situations that are relevant to students. Unfortunately, what we often witness is an oversimplification of mathematical content. When teachers try to connect mathematics to students' everyday lives, they often do so in a superficial and inconsistent manner. For example, we have encountered cases where teachers try to contextualize weight measurements by asking students to use them only when following a recipe for a class project. It is clear that there is no in-depth exploration of the mathematical concepts involved, nor any expansion of the practical applications of weight measurements.

Guiding principles

The keys to contextualizing mathematics are based on principles laid out in what is known as realist mathematics education. This approach, developed by Freudenthal, Gravemeijer, Puig, and Goffre, emphasizes viewing mathematics as a human activity. According to this perspective, teaching mathematics involves creating a connection between mathematical concepts and the student's real-world experiences. The goal is for students to view mathematics as a tool for organizing, understanding, and transforming the world around them.

From an epistemological point of view, this means moving from the current approach of presenting mathematics as a discipline with a fixed and unattainable deductive system, to a vision of mathematics as a continuous construction process. In this new approach, the interaction of students with their environment becomes a process of reinvention guided by the teacher. This type of contextualization extends beyond the human body to other phenomena or contexts (f1, f2, f3,...). For example, architecture and the visual arts provide ample examples of symmetry and rotation that can be studied using mathematical concepts. By incorporating these mathematical contents into the study of different contexts, the horizon of mathematical understanding is broadened.

This approach to mathematics education involves two simultaneous processes: horizontal mathematization and vertical mathematization. Horizontal mathematization is relating a set of non-mathematical situations to mathematical concepts. For example, when we are presented with the human body, we may not immediately see the mathematics that can be derived from it. However, mathematics can help us better understand the human body and vice versa. It is the teacher's role to discover and highlight the mathematical elements inherent to the human body. In this way, both mathematics and the human body are perceived in a new light, the so-called Didactic Phenomenology. In the case of the example of the human body, mathematics allows us to explore concepts such as symmetry, rotation, proportions and the golden ratio, using both conventional and non-conventional length measurement systems.

At the same time, while we focus on the process of horizontal mathematization in the classroom, it is important to address vertical mathematization as well. Vertical mathematization involves broadening the mathematical processes that are derived from the examination of various phenomena. In other words, it is not only about broadening the scope of different phenomena or contexts in relation to horizontal mathematization processes; but it also involves going deeper into the study of mathematical objects. This concept is known as vertical mathematization, as put forward by Treffers (1987). To better illustrate this, let us consider the example of the study of the human body. By exploring the human body, we can explore the interconnection of different mathematical objects and gain a deeper understanding of them. For example, by examining measurements and proportions, we can go deeper into the generalization of unit conversion and explore Thales' theorem.

The keys

The task of bringing the phrase "mathematics is everywhere" to life in the classroom is not easy, as they point out. Typically, there is a lack of diversity in the phenomena or contexts through which we teach mathematics, often relying on examples related to basic arithmetic and geometry. This situation can be attributed to differences in social practices and codes within the school system compared to those outside it.

The key, therefore, is to find ways to incorporate into our educational practices the social practices and communicative codes of mathematics that exist beyond the school environment. To achieve this incorporation of the mathematical world into our classrooms, there are various areas and people that can be approached. In this discussion, we will focus specifically on the role of the teacher.

An important aspect of teaching mathematics in a contextualized manner is for the mathematics educator to have a deep understanding of the mathematical concept itself, including its foundations, history, and real-world applications. By knowing the origins of a mathematical topic, educators can understand the issues that led to its development and find similar situations that can be adapted for learning purposes. For example, the discovery of the number pi by ancient civilizations such as the Greeks and Egyptians, who used it in measurement problems involving circles, can be replicated today if students manipulate circular objects to arrive at the same conclusions. In addition to historical context, it is also crucial to understand modern applications of mathematical concepts.

Educators should ask questions about the usefulness and relevance of the topic in different contexts and the problems it can solve. It is also important to recognize the connection between the problems being studied and the conceptual structure of the mathematical concept being taught. This involves identifying a set of situations that share a common underlying theme, which may be natural, social, or cultural. For example, when teaching about pi, educators can present relatable situations to students, such as designing flowerpots at school or solving industrial design problems that require choosing between a cylindrical container or a parallelepiped based on their respective capacities. In such cases, the value of pi plays a fundamental role in making the decision.

Another important aspect of contextualization involves the ability to search for information and analyze it from a classroom perspective. To understand the origins and applications of certain mathematical topics, it is necessary to explore different sources of information. One valuable source is information technologies, which provide access to a wealth of information through web 2.0 platforms. This wealth of information removes the excuse of not having knowledge about a topic.

Teachers must actively seek out these sources and, more importantly, learn to discern which ones are dependable and relevant to the specific situation at hand. This emphasizes the importance of being able to analyze the information provided to us.

Students can also play a role in this search for information, as they can be guided to become allies in this research. However, there are other sources that are often overlooked and neglected in the current educational landscape, such as bibliographic sources and key informants.

Key informants are people who possess specialized knowledge and insights but are rarely considered by schools. They could be instrumental in incorporating mathematical knowledge used outside the school context. For example, if we want to teach students the concept and calculation of area, we can find numerous resources on web 2.0 platforms and in school textbooks that provide formal knowledge on the topic. However, we have a limited understanding of how bricklayers, engineers, and architects use this mathematical concept in their professions. Their approaches may not always align with what is taught in school, but their methods are validated in their daily practice. So, why not invite them to share their knowledge and experiences with students? The goal is not to replace institutional knowledge with mathematical knowledge derived from social practices, but rather to complement both worlds and achieve a better understanding of their potentialities and limitations. By doing so, we can bridge the gap between mathematics taught in school and mathematics used outside of school.

To utterly understand students and connect with them on a deeper level, engaging in meaningful conversations and establishing emotional connections is essential. This involves not only seeking insights from experts and established knowledge, but also actively communicating with our students to learn about their interests, needs, and preferred methods of communication. By fostering this teacher-student connection, we can identify which contexts are most conducive to integrating mathematics into our students' lives. However, this requires the teacher to possess strong communication skills and a willingness to engage in dialogue.

Creating an environment that supports dialogue is crucial to its development. This means going beyond the traditional classroom dynamic and allowing for the presentation and discussion of mathematical ideas. Classrooms should be transformed into forums where everyone can freely express their opinions and share their findings on mathematical learning situations. However, the role of the teacher extends beyond the confines of the classroom. It is important for them to actively participate in other spaces within the school or even outside of it, where they can engage in honest and ongoing conversations with their students, the educational community, and society as a whole. In this way, we can gain a comprehensive understanding of the people we teach, including their cognitive abilities and the environment in which they operate, such as their social background, family dynamics, and community influences.

Teaching mathematics in a way that relates to students' everyday lives involves integrating its concepts into real-life situations. This means that the content taught in our classrooms must have practical meaning for students. From our perspective, we believe that this approach to teaching mathematics can help shape students into citizens who understand and transform the world in which they live, all within a framework of respect and freedom.

We strongly believe that contextualization is valid and relevant in today's educational system, but it is crucial to note that this contextualization should not be arbitrary. In order to effectively contextualize mathematics, there are three key factors to consider. First, the teacher must have a deep understanding of mathematical concepts, their origins, and their applications. Second, the teacher must be aware of the interests, needs, and context in which his or her students typically operate. Finally, the teacher must possess the ability to seek and analyze information in order to expand his or her own knowledge of mathematics, including its foundations and applications. By doing so, the teacher can create learning situations in which mathematics becomes a tool to explain the realities of students' lives, both present and future. School mathematics should go beyond the traditional image of a set of deductively defined theories and rules that remain unchanged over time. Rather, it should be seen as a constantly developing subject, where the educator's guidance is crucial.

Teachers must constantly reinvent their lessons in collaboration with students, exploring diverse contexts that contribute to a deeper understanding of the mathematical concepts being studied. The goal is for students to learn mathematics by actively participating in mathematical activities, and for this to happen, the content being taught must be meaningful and relevant.

All of these ideas presented here raise important questions about the training and professional development of mathematics teachers. These questions challenge us to consider what mathematics future teachers should learn, as well as what theoretical and methodological tools they should acquire to recognize mathematics in different contexts and design learning situations accordingly. These questions, among others, could be the focus of future research. As teacher educators, it is our responsibility to address these challenges and contribute to rectifying the long-standing gap between teachers and students, which has been perpetuated by teaching mathematics in isolation from real-world contexts.

The didactic perspectives of mathematics

It should be noted that there is a distinction between education and didactics. Education covers a broader scope than didactics, allowing us to differentiate between Mathematics Education and Mathematics Didactics. By adopting this approach, mathematics education is defined as "the whole system of knowledge, institutions, training plans and training purposes" that constitute a complex and diverse social activity related to the teaching and learning of mathematics.

Mathematics Education refers to the discipline that studies and investigates the challenges that arise in mathematics education and proposes well-founded actions for its transformation. However, in the English-speaking world, the term "Mathematics Education" is used to refer to the field of knowledge known as Mathematics Education in countries such as France, Germany and Spain. Mathematics Education is also identified as a scientific discipline and an interactive social system that encompasses theory, development and practice.

In his diagram, Steiner (1990) represents Mathematics Education (ME) as a discipline that is connected, as part of it, to another complex social system called the Mathematics Teaching System (MTS) - which Steiner refers to as "Education, Mathematics and Teaching", which is represented as the thicker circle outside of MTS. Within this system exist several subsystems, including the mathematics classroom (MC) itself, teacher education (TE), curriculum development (CD) and Mathematics Education (ME) as an institution that is part of MTS. Steiner further extends the diagram by including the whole social system concerned with the communication of mathematics, which encompasses new areas of interest for Mathematics Education, such as the issue of "new learning in society" (NS) brought about by the use of computers as a medium for teaching mathematical ideas and skills outside the school context.

It also includes the study of the interrelations between Mathematics Education and Experimental Science Education (ECE) within this field. Steiner considers theorizing activity (TEM) as a component of mathematics education and, therefore, of the broader system we refer to as SEM, which constitutes the system of teaching mathematics. TEM is positioned externally to consider and analyze the integral global system as a whole.

Another model proposing the relationships between Mathematics Education and other disciplines is that presented by Higginson (1980), who identifies mathematics, psychology, sociology and philosophy as the four fundamental disciplines. Higginson visualizes Mathematics Education in terms of the interactions between these four disciplines, represented by the faces of a tetrahedron.

These different dimensions of mathematics education encompass the fundamental questions that arise in our field, such as what to teach (mathematics), why (philosophy), to whom and where (sociology), and when and how (psychology). In Higginson's work, she also explores the applications of this model to clarify essential issues such as understanding traditional perspectives on mathematics teaching and learning, understanding the factors that have led to curricular changes in the past and predicting future changes, and examining the evolution of conceptions about research and teacher education.

The study of epistemological currents reveals that scientific theories cannot exist in isolation or be the product of individual efforts. Instead, there must be a community of researchers who share common interests and agree on appropriate methods for addressing research problems. It is important to strike a balance between personal autonomy in developing new ideas and the need for these ideas to be shared and tested within a community. Theories are therefore the result of collaborative research efforts within a specific field.

For a field of research to be considered “normal science” according to Kuhn’s criteria, certain conditions must be met:

- First, there must be a group of researchers who share common interests and focus on studying the interrelationships between different aspects of a complex real-world phenomenon.
- Second, the explanations provided by the theory must be causal and allow predictions to be made about the phenomenon.
- Finally, the group of researchers must agree on a common vocabulary, syntax, and procedures for testing and evaluating the theory.

Scientific concepts, propositions and theories are distinguished from non-scientific constructs by their adherence to the scientific method and logical reasoning, as well as by their acceptance by the scientific community. However, the requirement of a single paradigm or a unified community of specialists, as defined by Kuhn, may be too restrictive. In the social and human sciences, including Mathematics Education, it is natural and beneficial to have competing schools of thought, as they encourage the development of diverse research strategies and the exploration of problems from different perspectives.

The complexity of the phenomena studied may require the coexistence of multiple research programs, each supported by different paradigms taken from various disciplines. Bunge's (1985) epistemological approach, which considers scientific fields as sets of competing lines of research, seems more appropriate to understand the current state of Mathematics Education.

Some authors have categorized certain didactics as mere technical knowledge or, at most, technological knowledge, rather than recognizing them as educational sciences in their own right. However, when considering the relationship between general theory and specific theory, as Bunge explains, it becomes clear that special didactics are not simply subfields or chapters within general didactics or educational psychology. Rather, they represent specific theories that cover particular aspects of the broader field.

Each specific theory includes the general theory and subsidiary hypotheses that describe the unique characteristics of the objects being studied. While it is commonly assumed that the general theory includes all specific theories, Bunge argues that it is actually the other way around. The general theory can be derived from specific theories by eliminating the specific premises and focusing only on the assumptions common to all theories.

This distinction is important when considering the phenomena of learning and teaching. It is necessary to ask: learning from what? Teaching from what? The nature of the knowledge being taught, as well as psychopedagogical, social, and cultural factors, play a role in explaining and predicting learning and teaching phenomena. Therefore, the practice of Mathematics Education, including programming, curriculum development, and instructional strategies, must consider the specificity of the knowledge being taught.

The limitations of existing general educational theories result in the development of new theories that are better suited to explaining and predicting the phenomena they seek to understand. Indeed, these new theories may even introduce bold and innovative ideas that challenge the very foundations of established theories. The narrow framework of traditional teaching techniques, including the use of technology, is insufficient for theories that are built within certain branches of Mathematics Education research. Mathematicians, in contemplating the processes of creating and transmitting mathematical concepts, are forced to take on the role of epistemologists, psychologists, sociologists, and educators; in other words, they must also become didactic practitioners.

After carefully considering the criteria set by various authors to define a scientific discipline, we are left to ask whether the field of Mathematics Education meets these requirements. Specifically, we question whether there is a community of researchers within this field who are actively engaged in developing one or multiple research programs that can generate a comprehensive theory or theories of Mathematics Education. The following section aims to provide an overview of the current state of research in this area, with a particular focus on the contributions made by prominent research groups such as the Theory of Mathematics Education (TME) and Psychology of Mathematics Education (PME) groups. In addition, we will describe several key perspectives and approaches in the field, including problem solving and modeling, sociocultural frameworks, the French school of mathematics education, symbolic interactionism, the sociocritical point of view, and H. Freudenthal's didactic phenomenology.

Mathematics Education: Theory and Philosophy

During the 1990s, mathematics education research in the United States lacked a solid theoretical foundation and did not focus on building theoretical models. However, in the last two decades, there has been a significant change in this trend. Today, when publishing articles in peer-reviewed journals, it is mandatory to provide a clear reference to the theoretical framework that underpins the studies. This change is evident in the increasing number of publications that discuss and analyze various theoretical and philosophical approaches to mathematics education.

In 1984, Professor Steiner intended to form a research group at the 5th International Congress on Mathematics Education (ICME) that would focus on the development of a Theory of Mathematics Education. This led to the creation of a Thematic

Area called "Theory of Mathematics Education" at the Congress, which had four dedicated sessions. After the Congress, discussions continued at subsequent meetings and a Working Group called TME (Theory of Mathematics Education) was formed.

The TME conferences that have been held since then have shown that there is a community interested in building the theoretical foundations of Mathematics Education as a science. This community is made up of researchers in Mathematics Education, mathematicians, teachers, educational psychologists, educational sociologists and teacher trainers, among others. Steiner (1985) proposes that Mathematics Education should serve as a link between mathematics and society. This can be achieved by exploring forgotten dimensions of mathematics, such as the philosophical, historical, human, social and didactic dimensions.

By analysing the issues raised within the TME Group, which has attracted a majority of researchers interested in the theoretical foundations of Mathematics Education, we can begin to understand the central concepts of Mathematics Didactics as a scientific discipline. The formation of this scientific community is driven by professional interests and has fostered an academic orientation in its work. For example, in Germany, between 1960 and 1975, more than 100 professorships were created in teacher training colleges specifically for mathematics departments.

A similar phenomenon has been taking place in Spain since 1985 with the recognition of Mathematics Education as an area of knowledge and the creation of university departments for teachers in this area. However, there is a risk that Mathematics Education will become disconnected from social reality due to this academic approach. In general, the TME Group and its conferences aim to advance the theoretical foundations of Mathematics Education and promote its integration with other disciplines, while considering its practical implications and social relevance.

To address these components, the TME Group explores various topics such as the definition of Mathematics Education as a discipline, the use of models and theories in research, the role of macromodels and micromodels, the debate between specific theories and interdisciplinary approaches, the relationships between Mathematics Education and its reference fields, and the ethical, social and political aspects of Mathematics Education.

The group also emphasizes the importance of systems theory, particularly social systems theories, in understanding Mathematics Education as an interactive system. The TME Group's development program focuses on the current state and future prospects of Mathematics Education as an academic field and as an intersection between research, development and practice. This program consists of three components:

- (A) identify and formulate fundamental problems in the orientation, foundation, methodology and organization of Mathematics Education,
- (B) develop a comprehensive approach to Mathematics Education as an interactive system, and
- (C) examine the interdependencies and conditions in Mathematics Education, including the analysis of complementarities.

The Second Conference of the TME Group, which took place in 1985 at the Institut fur Didaktik der Mathematik (IDM) at Bielefeld University, focused on the broader topic of "Foundations and Methodology of the Discipline Mathematics Education (Didactics) of Mathematics." This conference primarily highlighted the role of theory and theorizing in specific domains within Mathematics Education. Some of the specific topics discussed included theories about teaching, the theory of teaching situations, interactionist theory of learning and teaching, the role of metaphors in developmental theory, empirical theories in mathematics teaching, fundamental mathematical theories, theoretical concepts for the teaching of applied mathematics, representation theory for understanding mathematical learning, and historical studies on the theoretical development of mathematics education as a discipline.

The conference working groups were engaged in analysing the use of models, methods, theories, paradigms and other research tools within different research domains. Despite the diversity of topics discussed at the TME conferences, there is still no clear consensus on the central issues and fundamental concepts within Mathematics Education. While the conferences have generated many partial results and practical guidance for the classroom, progress towards establishing a cohesive academic discipline with its own theoretical foundations is still lacking.

The theme of the Third Conference, held in 1988 in Antwerp, Belgium, focused on the role and implications of research in Mathematics Education for teacher education. This conference aimed to address the significant gap that exists between teaching and learning in this field. Some of the specific issues discussed included the gap between teaching and learning in mathematics classrooms, the gap between research on teaching and research on learning, models for teaching design based on research on learning, the need for theory and research in development work and projects, the role of content and different perspectives on mathematics in bridging the gap between research and learning, the role of social interaction in the classroom, and the implications of the conference theme for teacher education. In addition, the conference explored the role of computers as a third component in teaching-learning interaction.

The fourth Conference, held in Oaxtepec, Mexico in 1990, focused on two main themes: the relationships between theoretical orientations and empirical research methods in Mathematics Education, and the role of holistic and systemic aspects and approaches in Mathematics Education. This conference also marked the beginning of the presentation of various training programmes for researchers in Mathematics Education at different universities, both at the PhD and master's level. As part of this initiative, a questionnaire was distributed to universities around the world to collect information on research training, with the aim of establishing a network for the exchange of information and discussion on the topic.

The organization of research in Mathematics Education is a discipline that fulfills two main purposes:

- First, it provides information and data on the current status, problems and needs of Mathematics Education, considering national and regional differences.
- Secondly, it contributes to the development of meta-knowledge and a self-reflective attitude, which serves as a basis for the establishment and implementation of development programs in Mathematics Education.

At the fifth Conference, held in 1991 in Paderno del Grappa, Italy, a preliminary report on the results of the survey on the training of researchers was presented. In addition, a number of papers were presented on the role of metaphors and metonymies in mathematics, mathematics education and in the mathematics classroom, as well as the role of social interaction and knowledge development from a Vygotsky perspective. These conferences demonstrate the wide range of topics studied within the field of mathematics education, including mathematics itself, curriculum design, students' construction of mathematical meaning, teacher-student interactions, teacher preparation and alternative research methods.

The aims of this network (TME) include exploring current developments in the philosophy of mathematics, such as Lakatos' fallibilism, and other humanistic perspectives. They also aim to delve deeper into the philosophical aspects of mathematics education, ensuring that philosophical reflection receives the same consideration as other disciplines in this field. Furthermore, it aims to establish an open international network of people interested in these topics and to provide opportunities for the exchange and advancement of ideas and perspectives. The network seeks to foster informal communication, dialogue and international cooperation between teachers, researchers and others involved in theoretical and philosophical research on mathematics and mathematics education.

Interest in the theoretical and philosophical foundations of mathematics education has grown significantly since 2005, particularly after a research forum dedicated to this topic was held at the PME Annual Meeting in Melbourne. Since then, numerous researchers have published various papers in the ZDM journal and the topic has also attracted attention in one of the working groups of CERME (European Congress on Research in Mathematics Education).

This growing recognition of interest in mathematics education theory can also be seen in research handbooks in the field. For example, Silver and Herbst (2007) provide an overview of the state of theory in mathematics education research in the "Second Handbook of Research on Mathematics Teaching and Learning" edited by Lester (2007). Furthermore, Coob (2007) explores the topic "Putting Philosophy to Work: Confronting Multiple Theoretical Perspectives" in the same handbook.

The importance of theory development is also highlighted in the first and second editions of English's "Handbook of International Research in Mathematics Education" (2002) and (2008), respectively. These efforts have culminated in the publication of "Theories of Mathematics Education. Searching for New Frontiers" edited by Sriramn and English (Springer, 2010), which contains 19 main chapters along with prefaces and commentaries prepared by various authors. Topics covered in this book include perspectives on theories and philosophies of mathematics education, reflections on learning theories, theoretical and philosophical foundations of mathematics education research, the plurality of mathematics education theories, the reconceptualization of mathematics education as a design science, the fundamental cycle of concept construction underlying different theoretical frameworks, symbols, and mediations in mathematics education, and more.

The importance of constructing theories is evident, as they serve as a guide to formulating research problems and interpreting their findings. Theoretical frameworks allow the organization of knowledge within a specific field, which is an initial step towards a comprehensive understanding of the connections that exist in our perceptions. Theorization is a prerequisite for an area of knowledge to achieve scientific status and fulfill its function of explaining and predicting phenomena. Indeed, it can be argued that significant scientific research is always guided by a theory, even if it is not explicitly stated.

As Mosterín (1987) suggests, theories allow us to bring conceptual order to the chaotic and formless world, allowing us to reduce complexity to a formula. They provide us with tools for extrapolation, explanation, and a means of understanding and exerting control over the world, even if this understanding and control is always uncertain and problematic.

Mosterín also offers a compelling metaphor, comparing theories to the spider webs that we, like spiders, use to capture and make sense of the world. These webs should not be confused with reality itself, but without them, how much further away would we be from being able to grasp and appreciate the world around us? According to Lester (2010), employing a theoretical framework to conceptualize and guide research offers several important advantages:

- A framework provides a structure for conceptualizing and designing research studies. Specifically, it helps determine the nature of the questions being asked, the formulation of these questions, the definition of concepts, constructs, and processes within the research, as well as acceptable research methods for discovering and justifying new knowledge about the topic being studied.
- Without a framework, data is meaningless. Whether a data set can be considered evidence of something is determined by assumptions, beliefs, and the context in which the data was collected. An important aspect of a researcher's beliefs is the framework, whether theory-based or not, that they are using. This framework enables interpretation of the data set.
- A solid framework allows us to go beyond common sense. Deep understanding, derived from a commitment to theory building, is often crucial to addressing truly important problems.
- The goal is to achieve deep understanding. As researchers, we should strive to gain a comprehensive understanding of the phenomena we are studying, focusing on important questions rather than seeking only solutions to immediate problems and dilemmas.

The research framework helps to develop this deep understanding by providing a structure for designing research studies, interpreting the resulting data, and drawing conclusions. Lester (2010) distinguishes between three types of research frameworks:

- Theoretical frameworks, which guide research activities based on a formal theory that offers a coherent and established explanation of certain phenomena and relationships. Examples of relevant theories used in the study of learning include

Piaget's theory of intellectual development and Vygotsky's theory of sociohistorical constructivism.

- Practical frameworks, which are based on the practical knowledge accumulated by practitioners and managers, previous research findings, and often insights from public opinion. These frameworks guide research based on what has been shown to work in practice. Research questions are derived from this knowledge base, and research findings are used to support, extend, or revise existing practices.
- Conceptual frameworks, which are local theoretical models that justify the choice of concepts and their relationships in a particular research problem.

Like theoretical frameworks, conceptual frameworks are based on previous research, but are constructed using a variety of sources, both common and diverse. The framework used may draw on different theories and aspects of practical knowledge, depending on the researcher's argument about what is relevant and important to the research problem.

Burkhardt (1988) distinguishes between two types of theories: "phenomenological" theories and "fundamental theories." Phenomenological theories emerge directly from the data and provide a descriptive model of specific phenomena. They are characterized by limited applicability but are detailed and specific in their descriptions and predictions. They can be useful in curriculum design and in understanding phenomena because of their proximity to reality.

A fundamental type of theory is a conceptual framework that encompasses variables and their relationships, capturing the essential elements of a set of phenomena. It possesses both descriptive and predictive qualities and is comprehensive within a clearly defined domain. Such theories serve as analytical models with the aim of explaining a wide range of phenomena using a small number of fundamental concepts. The definition is particularly applicable to fields such as physics and biology, where theories such as Newtonian mechanics and Mendel's genetic theory align with this framework. However, when examining theories in the realm of human sciences, such as "behaviorism," "constructivism," and "developmental theories," Burkhardt raises questions about their nature and scope.

While these theories offer frameworks for understanding phenomena, they lack integrity within a limited domain. Consequently, they must be used with the understanding that they lack established mechanisms for reliable integration into a predictive model. Burkhardt regards them as overly simplistic descriptions of complex systems, which can potentially prove problematic. In the context of the physical sciences, these theories cannot be classified as comprehensive theories or even as models; rather, they are descriptions of "effects"—important aspects of a behavioral system that must be considered. However, each of these descriptions, on its own, is inadequate and can lead to misunderstandings.

The psychology of mathematics education

In the field of Mathematics Education, there is also a significant influence from a psychological perspective in the study of teaching and learning processes. However, this predominance of the psychological approach overlooks the importance of balance and complementarity with the other fundamental disciplines of Mathematics Education. This influence is evident through the prominence of the International PME Group (Psychology of Mathematics Education), which was established during the Second International Congress on Mathematics Education (ICME) and continues to hold annual meetings.

Among the main objectives of this group, as stated in its statutes, are to promote collaboration and the international exchange of scientific information related to the Psychology of Mathematics Education, to encourage interdisciplinary research in this area involving psychologists, mathematicians and mathematics teachers, and to deepen their knowledge of the psychological aspects of the teaching and learning of mathematics and its implications.

Review of the research reports presented at the PME annual meetings reveals that they encompass both empirical and theoretical research, covering a wide range of topics that extend beyond the strict boundaries of psychology. While it is not possible to provide a detailed account of the discussions held at these conferences due to their breadth, the classification scheme for the research reports is worth mentioning as it broadly represents current areas of focus within the field.

Cognitive interaction refers to instructional theories that emphasize the exchange of information between teachers and students, with the goal of facilitating students' assimilation of accurate information. This perspective includes theories proposed by Piaget, Bruner, and Ausubel, as well as those that highlight the interaction between instructional content and students' cognitive processes and skills. Social interaction, on the other hand, prioritizes the role of individuals involved in instruction as facilitators of learning. This perspective is represented by Vygotsky and Bandura.

Finally, contextual interaction theories, advocated by Skinner, Gagné, and Cronbach, among others, emphasize the interaction between individuals and contextual variables in the instructional process. Educational psychology is a field of study that focuses on the scientific examination of teaching and learning processes, as well as the challenges that may arise within these contexts. According to Gimeno Sacristán (1986), there are various perspectives that view teaching as a technique derived directly from a psychological theory of learning, which serves as its foundation. However, this dependence on psychology is considered detrimental to the development of a unique theoretical field for both General Didactics and Special Didactics, as it restricts their ability to create their own theories.

Consequently, educational psychology has the potential to dominate the study of human behavior in teaching situations, limiting the scope of Didactics. Within educational psychology there is a branch known as instructional psychology, which is defined as a "scientific and applied discipline that emerged from educational psychology and focuses on the study of psychological variables and their interaction with the components of the teaching and learning processes, taught by specific subjects, with the aim of teaching specific content or skills to equally specific individuals, within a specific context."

Researchers analyze and classify different instructional theories and models from an interactionist perspective into three types: cognitive, social, and contextual interaction. In considering the essential issues in Mathematics Education that can benefit from a psychological approach, Vergnaud (1988) identifies the analysis of students' behavior, their representations, and the unconscious phenomena that occur in their minds, as well as focusing on the behaviors, representations, and unconscious phenomena of teachers, parents, and other participants. In addition, he highlights four types of phenomena that can be fruitfully studied from a psychological perspective: the hierarchical organization of students' competencies and conceptions, the short-term evolution of concepts and skills in the classroom, social interactions and unconscious phenomena, and the identification of real theorems, schemata, and symbols.

Within the psychological approach, one of the key challenges is to identify theories about mathematical learning that can serve as a basis for teaching. Research on learning has provided limited insight into many central issues in instruction, and research on teaching often makes implicit assumptions about children's learning that are not consistent with current cognitive theories of learning.

Attempts have been made to apply general theories of learning to derive principles that can guide instruction. However, behaviorism-based instruction tends to fragment the curriculum into isolated parts that can be learned through reinforcement, which is not conducive to effective mathematics instruction that requires an understanding of fundamental mathematical concepts. Similarly, learning theories derived from Piaget's genetic epistemology have not adequately explained children's ability to learn mathematical concepts and skills.

This expansion of the field of interest of PME has led some, such as Fischbein (1990), to suggest that the psychology of mathematics education is becoming the paradigm for mathematics education as a whole. Fischbein argues that simply adopting issues, concepts, theories, and methodologies from general psychology has not yielded the expected results. He explains that psychology is not a deductive discipline, so the application of general principles to a specific domain does not usually lead to significant discoveries.

Even domains of psychology closely related to mathematics education, such as problem solving, memory, reasoning strategies, creativity, representation, and imagination, do not directly provide useful and practical recommendations for mathematics education and may not be the main source of problems in this field. Thus, the dynamics of mathematical symbolism requires a specific system of concepts beyond those inspired by general psychology.

Similarly, familiar psychological concepts take on new meanings in the context of mathematics and mathematics education. A fundamental assumption underlying current research on learning is derived from cognitive studies, which suggest that children actively construct knowledge through their interaction with the environment and the organization of their own mental constructs. While instruction certainly influences what a child learns, it does not determine his or her learning.

The child is an active participant in the process of knowledge acquisition, interpreting, structuring and assimilating information from his or her own mental frameworks. As Vergnaud points out, most psychologists interested in mathematics education today can be considered constructivists in some sense, since they believe that students themselves construct competencies and conceptions.

According to Kilpatrick (1987), the constructivist viewpoint involves two principles: knowledge is actively constructed by the learner and not passively received from the environment, and the process of knowledge acquisition organizes one's own experiential world rather than discovering an independent, pre-existing world external to the learner's mind. However, it should be noted that not all research in the field aligns with this perspective. In addition to the initial psychological problems faced by the PME group, the debate surrounding the research has highlighted the need to consider new aspects.

Two notable aspects include the specificity of mathematical knowledge and the social dimension. To study the learning of algebra, geometry or calculus it is necessary to carry out a deep epistemological analysis of the mathematical concepts involved. It is also important to recognize that the meaning of these concepts is not based solely on their formal definition, but rather on the processes involved in their operation. Therefore, attention should focus on studying students' cognitive processes rather than their current abilities or productions.

The social dimension is another crucial factor to consider in research on the psychology of mathematics education. The social status of the knowledge being taught and the role of social interactions in the teaching process require careful consideration. Moving from child-centered studies to studies centered on the student as a learner in the classroom is a significant step in the development of research in this field.

The student is a child engaged in a learning process within a specific environment, where social interactions with peers and the teacher play a vital role. This evolution of the research problem requires the development of more systematic classroom observations and the organization of specific teaching processes. It also requires the use of new theoretical and methodological tools to produce solid results that have both theoretical and practical significance.

However, the lack of specificity among researchers regarding the physical and social conditions under which knowledge is acquired allows for a wide range of epistemological viewpoints. These range from simple constructivism, which recognizes only one principle, to radical constructivism, which accepts both principles and denies the mind's ability to reflect objective aspects of reality. There is also social constructivism, which emphasizes the importance of cognitive conflict in the construction of objectivity.

According to Vergnaud, the solution to this epistemological dilemma is quite simple: knowledge construction involves gradually forming mental representations that are homomorphic to reality in some respects but not in others. From a methodological perspective, cognitive scientists observe individuals' problem-solving processes in detail, looking for patterns in their behavior and attempting to characterize these patterns with sufficient precision for students to use as guides for problem solving.

Its aim is to build "process models" of students' understanding, which are then tested using computer programs that simulate the solver's behavior. As mathematics educators, we must question whether the computer metaphor adequately explains the processes of teaching and learning mathematics and what implications information processing theories have for mathematics teaching.

Kilpatrick cautions against over-reliance on the information metaphor, reminding us that education should not be solely about transmitting information. While the information metaphor can be useful, it is important to recognize that there are different types of information and that something is lost when education is defined solely in terms of information acquisition. Some authors propose a different approach to problem-solving and teaching-learning processes, one that assigns a more active role to the solver and considers the specificities of mathematical content as well as the role of the solver.

When it comes to mathematics learning and information processing, there is currently no widely accepted theory that covers all the necessary details. Two main research approaches are identified in this field: constructivism, as mentioned above, and the cognitive science – information processing approach, which has had a significant impact on the study of mathematics learning.

Schoenfeld (1987) states that the underlying hypothesis of cognitive science is that mental structures and cognitive processes are complex but can be understood, leading to a better understanding of thinking and learning. The main goal is to explain what constitutes "productive thinking" or the ability to solve meaningful problems. Cognitive science uses the metaphor of the mind as a computer to understand cognition as information processing and, consequently, to understand teaching and learning processes.

The brain and mind are compared to the computer and its program, where cognition is carried out by a central processing mechanism controlled by an executive system that maintains awareness of its actions. Mental models are considered similar to general-purpose computer models with a central processor capable of storing and executing programs. In these models, the mind is considered unitary, with mental structures and operations invariant across different contents. A single mechanism is thought to underlie the ability to solve a particular class of problems.

Problem solving

Despite the attention paid to research on problem solving, there are doubts about its relevance to school practice. Some argue that teaching students problem-solving strategies and phases has little impact on their ability to solve general mathematical problems. This raises the question of why problem solving is so difficult for most people in mathematics. From our perspective, problem solving is not just a goal of mathematics teaching, but the essential means to achieve learning.

Students should have regular opportunities to engage in challenging problem-solving tasks, which will help them develop critical thinking skills, perseverance, curiosity, and confidence in unfamiliar situations. Problem-solving should be integrated into the mathematics curriculum and not treated as a separate component. It should be connected to the study of different mathematical content areas and incorporate contexts that are relevant to students' lives and other disciplines. However, there is a lack of studies exploring the conceptual development that arises from problem-solving and its interaction with the development of problem-solving competencies.

Problem solving has emerged in recent years as an important area of research in mathematics education. This research was initially spurred by the influential work of Polya in 1945, which led to a large body of research on topics such as computer-simulated problem solving, expert problem solving, strategies, heuristics, metacognitive processes, and problem posing. More recently, there has been an increasing emphasis on mathematical modeling in the elementary and secondary school grades, as well as on interdisciplinary problem solving. Many of the early studies focused on typical word problems found in school textbooks and tests.

These problems may be routine, requiring standard computational methods, or nonroutine, involving finding a solution when the path is not obvious. Nonroutine problems are particularly challenging for students. The importance given to problem solving in the curriculum and educational research arises from the belief that problem solving is at the core of mathematics. Authors such as Lakatos and Polya have contributed to this perspective, and Polya describes four phases of problem solving: understanding the problem, conceiving a plan, executing the plan, and examining the solution.

Polya's book has been highly regarded by mathematics educators as a valuable resource for improving students' nonroutine critical thinking skills and addressing the common question of what to do when stuck on a problem. However, while Polya's work describes the ideal analytical person, Schoenfeld's research focuses on the actual behavior of real problem solvers. Schoenfeld suggests that problem-solving instruction should help students develop a repertoire of strategies specific to different types of problems, promote metacognitive strategies for self-regulation, and work to improve students' beliefs about mathematics and problem solving.

The sociocultural perspectives discussed by Sierpinska and Lerman highlight the importance of considering the social and cultural context in mathematics education research. Understanding the role of social factors, the mediation of tools, and the development of consciousness can contribute to a more comprehensive and effective approach to mathematics teaching and learning. One research approach that has been developed along these lines is Activity Theory. This theory emphasizes the role of the acting person and the mediation of meaning between the individual and the world. For the child, society and culture are mediated through tools, particularly cultural tools.

Thought and language are considered dialectically related, as language provides the child with inherited historical-cultural meanings, but these meanings are continually reconfigured through intersubjective communication and action. In recent years, there has been a growing interest in the social context of the mathematics classroom in mathematics education research.

The role of social context in the development of individuals or groups has been theorized in a variety of ways. However, the current focus has shifted from identifying social factors in the affective domain to understanding the impact of the broader social and cultural environment on child development. In their 1996 review of epistemologies in mathematics education, Sierpinska and Lerman discuss sociocultural views that have been applied to the field of research. The term "sociocultural" refers to epistemologies that view individuals as situated within cultures and social situations, so that context and activity need to be considered when discussing knowledge or individuals.

Knowledge is seen as culturally produced, subject to change, and influenced by social values and regulations. Vygotsky and his followers, on the other hand, were primarily concerned with learning and teaching. Vygotsky did not delve deeply into the nature of mathematics or other forms of knowledge, except in psychology, which he sought to redefine as a materialist science. His main focus was the development of consciousness, which he believed was driven by communication and learning.

Vygotsky identified two types of thinking: ordinary or spontaneous thinking, which occurs informally through interactions with peers and adults, and scientific or theoretical thinking, which consciously aims at teaching and learning through the child's appropriation of cultural knowledge. An important aspect of Vygotsky's approach is the recognition that individuals and the world they inhabit are products of their time and place. An individual's psychology, expressed as consciousness, is shaped by the mediation of tools, which are influenced by social, historical and cultural context.

This perspective challenges Cartesian dualism and emphasizes the interconnectedness of subject and object. Based on this understanding, it is argued that there is no parallel between epistemological obstacles in mathematics and cognitive obstacles in learning. For example, the concept of negative numbers faced epistemological obstacles in the development of mathematics in the West, but today children can learn about negative numbers without recreating that historical struggle.

This suggests that there is no inherent reason to assume a similar parallel between epistemological and cognitive obstacles. Vygotsky introduced the concept of the zone of proximal development, which refers to the difference between what a child can do independently and what he or she can achieve with the help of a peer or experienced adult. This concept highlights the importance of learning with others and suggests that learning leads to development.

The perspective contradicts Piaget's belief that development, represented by stages of child development, drives learning. Vygotsky also emphasized the process of internalization, which involves the formation of consciousness through the mediation of tools that are expressions of the social, historical, and cultural situation. This view integrates teaching and learning at the school level.

Lave introduced the concept of knowledge in action, which contrasts with a cognitive perspective and emphasizes the role of context in mathematical practices. His studies focused primarily on the application of mathematical skills in everyday life and work situations. Lave criticized the traditional approach to school mathematics, which prioritizes generalizable techniques and skills, arguing that it should be more relevant to everyday life.

The concept of socioepistemology is used primarily in the Latin American educational mathematics community. It is a theoretical framework that suggests examining the production and dissemination of mathematical knowledge from different perspectives. This framework originated from research conducted by Cantoral, Farfán and other academics from the Higher Education Section of the Department of Educational Mathematics at CINVESTAV (IPN, Mexico).

Socioepistemology not only offers an expanded understanding of epistemology, emphasizing the socio-epistemic relativity of the meanings of mathematical objects in line with other sociocultural viewpoints, but also provides a systematic approach to studying the interactions between this mathematical understanding and cognitive and instructional aspects. It proposes the examination of mathematical knowledge considering its social, historical and cultural context, exploring how it was constructed and disseminated. In addition to recognizing problem solving as a fundamental aspect of mathematics, the need to explain the sociocultural factors involved in the construction of mathematical knowledge, the role of the tools used and the various interpretations of mathematical objects is also recognized.

Current research in Mathematics Education significantly focuses on the idea that teaching and learning processes should aim to empower individuals and achieve social transformation. To achieve this, it is necessary to promote strategies that encourage reflection on practice by the individuals involved, which can lead to significant changes in teaching approaches. An example of a research program that aligns with this perspective is known as "Critical Mathematics Education." This approach presents an agenda for studying the relationship between mathematics education and democracy. Some of the key aspects emphasized by critical theory include:

- preparing students to be active citizens;
- use mathematics as a tool to critically analyze socially relevant issues;
- consider students' interests and perspectives;
- consider cultural conflicts that may arise in the instruction process;
- draw on prior experiences in teaching and learning mathematics to develop critical thinking skills; and
- give importance to communication within the classroom since it forms the basis of democratic interactions.

Another area of concern within critical mathematics education is the intersection between mathematics and technology, which, while solving problems, also creates new challenges. From a socio-critical perspective, teachers are encouraged to shift their role from mere facilitators to active constructors of knowledge. It is argued that teachers have the capacity and should be involved in developing pedagogical theory based on educational research, bridging the gap that has traditionally separated theory and practice, where theory is usually left to researchers and practice to teachers in their daily work.

The researcher becomes an actor committed to achieving change. Participatory action research is often used as a research methodology in this context. Action research, when applied in the school setting, involves studying a social situation in which teachers and students actively participate in order to improve the quality of their actions. This is done through a cyclical process of problem identification, planning, implementation, reflection and evaluation of the results.

Increasing attention has been paid to using semiotics, the "science of signs", to describe and understand the teaching and learning of mathematics. This interest is driven by several factors:

- First, there is a growing recognition that mathematical activity is fundamentally a symbolic activity because of the generality of mathematical objects.
- Second, understanding classroom communication has emphasized the importance of understanding the nature of mathematical discourse for researchers and teachers.
- Semiotics provides an adequate theory to account for the complexity of communication.
- Furthermore, the increasing use of technology in mathematics education has led to the exploration of semiotics as a means of understanding the cognitive role of artifacts.
- Semiotics is well suited to this task because of its focus on the cultural conventions and meanings associated with signs and artifacts.

The unique insights that a semiotic perspective brings to the understanding of communication and learning in mathematics aim to model the role of mathematical sign systems, meaning structures, mathematical rules, and the motivations behind mathematical activity within a coherent framework. The use of semiotics in the study of mathematical activity is justified given the essential role of signs in mathematics.

Signs, symbols and notations play a similar role in communicating mathematical ideas in both educational contexts and learning processes. The semiotic perspective differs from psychological perspectives by focusing on signs and their use rather than solely on mental structures and functions. It encompasses both the individual and social dimensions of mathematical activity, teaching and learning by considering mathematics as a communicative act.

Semiotic systems, consisting of signs, rules of sign production, and the relationships between signs and meanings, are considered integral to understanding the use of signs in mathematics. Godino and his collaborators have developed an "ontosemiotic approach" to mathematics education that recognizes the fundamental role of language, semiotics, and ontological issues in describing and understanding the processes of teaching and learning mathematics. They view mathematical objects as emerging from the systems of practices used to solve specific problems, and this perspective complements existing semiotic perspectives in mathematics education.

A significant body of research in the field of mathematics education focuses on examining the connections between teachers, students, and mathematical tasks in mathematics classes. The goal is to find well-founded answers to questions such as how teachers and students develop a shared understanding of mathematical concepts to ensure a smooth flow of the class.

Researchers also investigate how students understand and respond to teacher interventions. To address these questions, it is essential to develop theoretical perspectives that can effectively interpret and analyze the intricate nature of mathematics lessons. Some of the key questions that interactionism addresses in mathematics education include: how mathematical meanings are interactively formed in different classroom cultures, how these meanings are stabilized, and how they are influenced by the type of classroom culture in which they evolve.

The interactionist program introduces concepts such as domains of subjective experience, patterns of interaction, and sociomathematical norms. The notion of domains of subjective experience, developed by Bauersfeld, Krummheuer, and Voigt (1988), adapts psychological concepts such as "script," "frame," "expert system," and "microworld" to the study of mathematical learning. According to this model, individuals form experiences within specific contexts and situations, incorporating cognitive, emotional, and motor aspects. These experiences are then stored in memory as distinct domains of subjective experience, reflecting the complexity and relevance of the situation as perceived by the individual.

Symbolic interactionism (SI) is a theoretical perspective that has been used to examine these relationships and has analytical implications. It asserts that cultural and social dimensions are not peripheral to mathematics learning but rather intrinsic to it. According to Sierpinska and Lerman (1996), who synthesized the interactionist program applied to mathematics education, interactionism is an approach that promotes a sociocultural understanding of the sources and development of knowledge.

The focus of study is on the interactions between individuals within a culture, with an emphasis on the subjective construction of knowledge through interaction. This perspective assumes that cultural and social processes are integral to mathematical activity. One approach, as suggested by Bauersfeld (1994), is to use theoretical constructs from sociology and linguistics, such as ethnomethodology, social interactionism, and discourse analysis. However, since these disciplines do not directly address the teaching and learning of curricular content, some translation is required to address the specific issues within mathematics education.

This approach is based on the premise that different practices emerge in the classroom depending on whether mathematics is seen as a collection of objective truths or as a process of shared mathematization. The latter perspective emphasizes the importance of the "interactive constitution" of meaning in classrooms, highlighting the relationships between the social characteristics of interaction processes and the thinking of both teachers and students.

The foundations of the interactionist perspective can be summarized as follows: teacher and students interactively shape classroom culture, conventions and agreements emerge through interactive processes, and communication is based on negotiation and shared meanings. The goals of research within the interactionist program in mathematics education, as stated by Sierpinska and Lerman (1996), are to achieve a better understanding of teaching and learning phenomena in ordinary school contexts. The main goal is not to develop theories for action or to design teaching actions, but rather to describe and discuss different possibilities. The research does not aim to improve the microculture of individual classrooms in the same way that it can influence the mathematics curriculum or the macroculture characterized by general principles and teaching strategies.

Negotiating the meaning of a particular situation can be fragile and prone to different interpretations due to ambiguity. Even if there is a shared context, there is always a risk of breakdown and disorganization during the interactive process. To minimize this risk, interaction patterns are formed. These patterns are considered regularities that are created through the interaction between teacher and students, seeking to make human interactions more predictable and less risky in their organization and development. Furthermore, interactions between teachers and students are often guided by implicit norms or obligations. From an interactionist perspective, the use of language is crucial, emphasizing the importance of negotiating meanings in the development of students' understanding of mathematical concepts and their beliefs and attitudes towards mathematics.

Chapter 4

Fundamental didactics

In recent years, there has been a remarkable growth in interest and research around Mathematics Education. A group of researchers, including notable figures such as Brousseau, Chevallard and Vergnaud, have been working to develop a theoretical understanding of mathematics didactics. The approach, known as the "fundamental" conception of Didactics, distinguishes itself from other approaches by emphasizing a global view of teaching, a strong connection to mathematics, specific learning theories, and a search for unique research paradigms.

This line of research aims to establish an original theoretical framework, developing its own concepts and methods, and considering teaching-learning situations in a comprehensive manner. The investigation of these issues requires a methodological approach that involves experimentation in a dialectical interaction with theory. Experimental observations are compared with the theoretical framework and adjustments can be made based on the coherence of the concepts developed and their comprehensiveness in relation to the relevant phenomena.

The Mathematics Didactics approach is based on a systemic vision, considering the overall functioning of teaching-learning phenomena. It recognizes that the separate study of individual components cannot fully explain the overall functioning, just as it cannot explain economic or social phenomena. Chevallard and Johsua describe the Didactic System as being composed of three main subsystems: the teacher, the student, and the knowledge taught. In addition, the system is influenced by the world outside the school, including society, parents, and mathematicians. The intermediate zone, known as the noosphere, is a place of conflict and transactions that facilitates the articulation between the system and its environment. It encompasses all individuals in society who reflect on the content and methods of teaching. The media, which consist of the materials, games, and teaching situations with which the student interacts, are also included as a component.

This line of research in Mathematics Education seeks to understand the production and communication of mathematical knowledge, focusing on the specific characteristics of these processes. It considers teaching and learning phenomena from a systemic perspective and emphasizes the interaction between the teacher, the student and the knowledge taught. The development of an original theoretical framework, the use of experimentation alongside theory and the exploration of diverse concepts and methods are key aspects of this approach.

The models that have been developed include exploration of epistemological, social, and cognitive dimensions. They strive to understand the complex interactions between knowledge, students, and teachers within the classroom context. One researcher, Laborde, has raised two important questions in relation to the study of teaching and learning in mathematics. First, how can the conditions for effective teaching be characterized to facilitate specific types of learning? And second, what elements should be included in the description of a teaching process to ensure that it can be replicated in terms of the learning it induces in students? These questions guide research and emphasize the importance of determining the mathematical knowledge that students wish to construct and comparing it to what is actually achieved during the teaching process.

The theory we are discussing encompasses its own perspective on mathematical learning, based on a Piagetian approach that emphasizes the construction of knowledge through the continuous interaction between the student and the subject matter. However, this theory is distinguished from other constructivist theories by its particular focus on the relationship between the student and knowledge. While content serves as the basis for the development of mental structures, the didactic point of view adds another layer of importance to the study of the student-knowledge relationship.

The main concern of research is the exploration of the conditions under which knowledge is formed, with the ultimate goal of optimizing, controlling, and reproducing it in educational settings. This requires paying particular attention to the object of interaction between the student and knowledge, i.e. the problem-solving situation, and how teachers manage this interaction. Recognition of the crucial role that situational aspects, context, and culture play in shaping students' cognitive behaviors is highlighted in the field of Mathematics Education Psychology, although this situational dimension is often overlooked as a separate area of research.

However, G. Brousseau's Theory of Didactic Situations stands as an initiative that addresses this gap. The relationship with knowledge is examined from a perspective of relativity, considering that knowledge can vary depending on the institutional context. For example, someone may be considered to have knowledge of probability within the scope of school education, but not within the academic sphere, and even within the academic world there are further distinctions based on the different levels of expertise required.

It is therefore necessary to differentiate between the institutional relationship to knowledge (what is considered acceptable within a particular institution) and the personal relationship to knowledge (an individual's understanding of a given topic), which may or may not align with the institutional perspective. Two fundamental questions arise from these concepts:

- What conditions ensure the successful integration of a specific element of knowledge and its institutional and personal relationships?
- What restrictions could hinder compliance with these conditions?

The study of the institutional relationship with knowledge, its conditions and its effects are considered the central problem of Didactics. While the study of personal relationships with knowledge is crucial in practice, it is considered epistemologically secondary. However, this study program cannot be successful without considering the various conditioning factors (cognitive, cultural, social, unconscious, physiological, etc.) that may influence or affect a student's personal relationship with the knowledge in question.

The relativity of knowledge within different institutions gives rise to the concept of didactic transposition, which refers to the process of adapting mathematical knowledge to make it suitable for teaching. In the initial phase of transposition, mathematical knowledge is transformed into pedagogical knowledge. This involves moving from describing the uses of a concept to describing the concept itself and the organizational advantages it offers. The process of didactic transposition involves decontextualizing the concept and removing its historical context, thus presenting it as a timeless reality detached from its origin, utility or relevance.

Once the concept is introduced, the didactic operation takes over, using it for educational purposes that do not necessarily align with the original intentions of its creators. As the concept is integrated into the knowledge taught, it undergoes a process of recontextualization. However, at the first educational levels, this recontextualization may not completely restore the original mode of existence of the concept or fulfill all the functions intended for its introduction.

To go deeper into the topic of conditional probability, it is worth mentioning that high school textbooks often introduce a concept called a "conditional event," which is not typically found in academic probability calculus. This concept refers to the event where B occurs given that A has already occurred, and is denoted as B/A. However, it is important to note that the event algebra is always isomorphic to a set algebra, meaning that the available operations are limited to union, intersection, and difference.

The study of didactic transposition focuses on identifying and analyzing these differences and understanding the reasons behind them, in order to rectify any misconceptions and ensure that mathematical objects are correctly understood in teaching. The brief description we have provided of some theoretical notions developed by French didacticians serves as an example of how the French School of Mathematics Didactics is establishing a solid foundation of theoretical concepts.

These concepts form the basis of a research programme similar to Lakatos' approach. The ability of researchers in this field to pose new research problems and offer new perspectives on classic problems is evident in their scientific production. Terms such as didactic transposition, didactic contract and obstacle are increasingly used in publications and international conferences focused on Mathematics Education. It is undeniable that France has a distinct line of research in this field, as demonstrated by Balachef's work, which represents an epistemological advance for this scientific discipline. It remains to be seen whether this line of research will end up becoming the predominant paradigm in the future.

Hans Freudenthal is an esteemed author in the field of mathematics education who has made important contributions to the topic. His book, "Didactic Phenomenology of Mathematical Structures," is widely regarded as a valuable resource for didactic research, curriculum development, and the practice of mathematics teaching. Two key concepts introduced by Freudenthal continue to generate interest and reflection: "didactic phenomenology" and the "constitution of mental objects."

Freudenthal criticizes the concept acquisition approach, which he believes views mathematics as conceptual structures separated from their cultural and problem-solving origins. In traditional teaching methods, the emphasis is on students learning mathematics as a finished product, devoid of its practical application. Freudenthal advocates prioritizing phenomenology, the problem situations that drive mathematical action, and the development of problem-solving strategies. These problem situations allow students to begin to constitute “mental objects,” which are personal cognitive structures that can then be enriched by a discursive and cultural understanding of mathematics.

The constitution of mental objects, as analysed by Freudenthal, challenges the conventional approach of trying to instil abstract mathematical concepts in students without providing concrete examples or experiences. Freudenthal argues that attempting to materialise bare concepts through concretisation often proves insufficient, as concretisations are often inadequate representations of the essential features of concepts.

Instead, Freudenthal suggests starting with phenomena that demand organization and teaching students how to manipulate the means of organization from that starting point. This approach reverses the traditional method of teaching abstractions by making them concrete. To implement this approach effectively, the assistance of didactic phenomenology is necessary to develop plans and strategies. Didactic phenomenology, as defined by Freudenthal, involves using mathematical concepts, structures, and ideas to organize phenomena both in the real world and in mathematics itself.

For example, geometric figures such as triangles, parallelograms, rhombuses, and squares help us organize boundary phenomena, while numbers organize quantity phenomena. At a higher level, geometric constructions and demonstrations organize the phenomenon of geometric figures, and the decimal system organizes the phenomenon of numbers. The phenomenology of a mathematical concept or structure, according to Freudenthal, involves describing its relationship to the phenomena it organizes, identifying the phenomena for which it was created and those to which it can be extended, understanding how it acts as a means of organization, and recognizing the power it gives us over those phenomena.

When the focus is on how this relationship is acquired in a teaching and learning process, we speak of didactic phenomenology of that concept or structure. Accordingly, the work of Hans Freudenthal highlights the importance of didactic phenomenology and the constitution of mental objects in mathematics education. By understanding the relationship between mathematical concepts and the phenomena they organize, and by starting with problem situations to develop cognitive structures, students can gain a deeper and more meaningful understanding of mathematics.

The field of research on mathematics teaching and curriculum in Mathematics Education is highly intriguing. At the practical level, curriculum and instruction play a central role in improving school mathematics programs and raise important research questions. By incorporating findings from other areas of Mathematics Education, particularly learning theories, research on curriculum and instruction aims to systematically understand and improve several aspects:

- the selection and organization of mathematical ideas to be taught;
- presenting these ideas to students; and

- evaluating program effectiveness and student performance.

It seeks to determine the most effective combinations of content, sequencing, strategies and delivery systems for different student skill profiles.

The complexity of research on curriculum and teaching is a notable feature. Consequently, designers of curriculum materials and instructional procedures often rely on personal creativity, intuitive judgments, and informal testing. Limited research is available explaining how the system can transform a combination of needs, interests, and values into a scientifically sound curriculum. As a result, topic selection in school mathematics is determined by factors such as the internal structure of the discipline (without rigorous epistemological analysis), public interest (measured informally), recommendations from respected experts, and sometimes textbooks prepared with little scientific basis.

Therefore, there is currently no consistent theoretical and experimental basis for research on curriculum and instruction. The search for a theory of instruction as a priority topic for future research merits designing theoretical models that establish relationships between key curricular and instructional variables. While the primary goal in this field has been to find the best method of instruction, efforts to identify general procedures, sequencing strategies, or appropriate presentation formats have been unproductive. Consequently, research now focuses on microscopic analyses of the curricular process and on exploring the expected effects of specific approaches in particular situations and content areas.

Another area of curriculum and instruction research investigates general questions independent of specific content. Thus, most research on teaching has not directly addressed mathematics, and the few studies that have focused on mathematics teaching have aimed to improve traditional methods rather than align with cognitive research perspectives. As a result, these studies may have irrelevant or potentially harmful findings.

In many research studies on teaching, the content being taught is often overlooked or considered peripheral. Therefore, the need for research that considers specific content and teaching techniques appropriate to that content is recognized. In general, studies conducted within the process-product paradigm for teaching mathematics have not provided teachers with a comprehensive list of observable behaviors that would enhance their competence and ensure student learning.

This reflects the early stages of what Kuhn (1969) called "normal science," where a paradigm or set of organizing principles that make all facts potentially relevant is lacking. Studies of mathematics teaching conducted under an interpretivist paradigm, although less common than positivist approaches, offer valuable insights into different aspects of mathematics teaching through different conceptual lenses. For example, research into a teacher's thinking about and teaching mathematics, and the impact of these beliefs on their teaching practices, is gaining increasing interest.

Is it simply a matter of practical knowledge, a technology that is based on and depends on other sciences, or is it that there are problems that require a level of theoretical analysis and a methodology proper to true scientific knowledge? This epistemological reflection is crucial to effectively guide didactic research, as it influences the formulation of its central questions. However, there has been limited discussion on this topic in the literature. The extreme complexity of the problems of Mathematics Education leads to two extreme reactions: those who claim that Mathematics Didactics cannot be based on scientific foundations and, therefore, teaching mathematics is essentially an art; and those who believe that Didactics can be a science, but only focus on a partial aspect of the problems, such as content analysis, curriculum construction, teaching methods, development of skills in students and classroom interaction.

This reductionist approach leads to different definitions and perspectives. Mathematics Didactics can be seen as the art of teaching: a set of means and procedures for making mathematics known. However, two scientific conceptions are distinguished, which are called the applied multidisciplinary conception and the autonomous conception (also called fundamental or mathematical). As a bridge between these two groups there is also a technical conception, which considers didactics as a teaching technique.

From the perspective of the multidisciplinary conception, which is aligned with Steiner's second tendency, didactics becomes a convenient label for the teachings necessary for the technical and professional training of teachers. Didactics, as a field of scientific knowledge, would imply research on teaching within established scientific disciplines such as psychology, semiotics, sociology, linguistics, epistemology, logic, neurophysiology, pedagogy, pediatrics and psychoanalysis. In this case, didactic knowledge would be a technology based on other sciences. The autonomous conception seeks to integrate all the aforementioned meanings and assign them a place in relation to a unifying theory of the didactic fact, with specific foundations and methods that point to an endogenous justification.

This conception can be the starting point to address the need for a theoretical basis that allows a better understanding and identification of the various positions, aspects and intentions underlying the different definitions of Mathematics Education, and to analyze the relationships between these positions in a dialectical understanding of the entire field. The French School of Didactics aims to build its own scientific field of study, which is not limited and dependent on the development of other scientific fields, which may not always be consistent.

This aim contrasts with the position of those who do not advocate the search for internal theories (household theories) because of the risk of inappropriate restrictions. The nature of the subject and its problems demand an interdisciplinary approach, and it is believed that it would be a mistake not to make significant use of the knowledge that other disciplines have already produced on specific aspects of those problems. Mathematics Education should strive for transdisciplinarity, as defined by Piaget, which encompasses not only interactions or reciprocities between specialized research projects, but also locates these relationships within a total system without fixed boundaries between disciplines.

The nature of mathematics education research is also considered, questioning whether mathematics educators should see themselves as applied educational psychologists, applied cognitive psychologists, or applied social scientists. Alternatively, should they be considered scientists in the field of physics or other pure sciences? Or should they be seen as engineers or other design-oriented scientists, whose research draws on multiple practical and disciplinary perspectives, guided by the need to solve real problems and develop relevant theories?

Brousseau's 1988 analysis examines how his conception of Mathematics Didactics, as a theory for communicating mathematical knowledge, compares with other perspectives and orientations. He argues that there is no conflict between his theory and others, but that his theory encourages the integration of ideas from different domains and their application to teaching. His theory promotes a healthy relationship between science and technology, rather than focusing on prescriptions and reproductions. Brousseau does not categorically condemn any educational action, but he warns against expecting didactics to fulfil functions that it does not have to fulfil.

He believes that it is a mistake to impose didactics on all educational action, as this can create challenges that may be beyond their capabilities. At worst, this can result in experts in the field taking on responsibilities for which they are not prepared, leading to errors similar to those seen in other disciplines such as economics. As Godino argued in 1990, the improvement of mathematics education depends on factors outside of didactic research itself, such as curricular guidelines, assessment procedures and teaching materials. Therefore, it is essential to facilitate communication between those responsible for these factors and researchers, as well as to promote didactic research. While didactic research cannot provide teachers with model situations to imitate, it can provide them with valuable knowledge to address the challenging nature of teaching mathematics in the classroom.

Paradigms

The fundamental or mathematical conception aims to integrate all the above-mentioned meanings and assign them a place in relation to a unifying theory of the didactic phenomenon. This theory would have specific and endogenous justifications and methods. This conception could potentially address the need highlighted by Steiner for a theoretical basis that would improve understanding and identify the various positions, aspects and intentions underlying the different definitions of mathematics education. It would also analyse the relationships between these positions and bring them together in a dialectical understanding of the entire field.

When mathematics educators or a group of teachers embark on research in their field, they are immediately faced with the epistemological problem of understanding the nature of Mathematics Education and the corresponding methodological paradigms. These issues influence the formulation of research problems and the determination of their significance. In our case, where a research tradition and established paradigms are lacking in the field, it becomes even more crucial to clarify the principles that have shaped Mathematics Education Theory and the potential research methods, as they dictate the types of research that can be conducted.

A literature review and synthesis by Hurford (2010) on the application of theoretical insights from complex and dynamic systems theory to understanding learning processes convincingly supports Steiner's views on the systems approach to mathematics education. Hurford suggests that educational researchers now have the tools and opportunity to build learning models that encompass inherent complexity in ways that were not previously feasible.

It is time to move beyond simplistic models that reduce learning to basic stimulus-response pairs or to static collections of isolated scenes of student learning. The perspectives and models offered by systems theory for understanding learning are preparing us to take that important step forward. The complexity of Mathematics Education is its defining characteristic.

As described by Steiner, mathematics encompasses the intricate phenomenon of mathematics in its historical and contemporary development, its interrelationship with other sciences, practical areas, technology and culture. It also encompasses the complex structure of teaching and schooling within our society, as well as the diverse conditions and factors that influence the cognitive and social development of students.

This complexity has led many authors to adopt a Systems Theory approach in their theoretical considerations. The interdisciplinary notion of system, which is embraced by all social sciences, becomes necessary when it is understood that the overall functioning of a set of elements cannot be explained solely by their individual contributions.

In fact, the behaviour of these elements can even be influenced by their inclusion in the system. In the case of mathematics teaching, a systemic approach is essential. It not only considers the mathematics teaching system as a whole and the conceptual systems that comprise it, but also considers the teaching systems that manifest themselves in a classroom.

The main subsystems in this context are the teacher, the students and the knowledge being taught. Adopting a systemic approach to teaching problems is fundamental because it highlights that Mathematics Didactics is at the centre of multiple interactions and, therefore, must develop its own problems and methodologies. However, this does not mean ignoring the contributions of related disciplines, particularly psychology and epistemology.

Furthermore, a systemic approach reveals the common structure that connects the didactics of various disciplines, but also recognizes the unique challenges posed by different domains of knowledge. Steiner further emphasizes that the systemic view of mathematics didactics is self-referential, as it includes mathematics education as one of its own subsystems. This self-referentiality necessitates a systemic approach as an organizational metaparadigm for mathematics education, not only to manage the complexity of the field as a whole but also because the systemic character is evident in each specific problem within the field.

From the discussion of these conceptions it emerges that there is a dialectical debate between the production of theoretical knowledge and practical knowledge in didactics. To clarify this distinction the labels "Theoretical Didactics" and "Technical (or

Practical) Didactics" can be used. The first refers to the academic discipline that aims to describe and explain the states and evolution of didactic and cognitive systems, while the second focuses on the problems of decision-making in the classroom and reflective action in specific contexts.

The theoretical perspective prioritizes understanding how the system works and discovering general laws that explain its dynamics, since the application of these principles can lead to the solution of specific problems. On the other hand, the practical perspective, adopted by researchers and applied professionals, recognizes the urgency of solving immediate problems without waiting for theoretical science to discover general principles. This theory-practice debate is not exclusive to Didactics but is observed in various sciences, including medicine and economics.

In Mathematics Education, both the technical and multidisciplinary conception adopt an applied science point of view, relying on general theoretical principles from other disciplines such as psychology, pedagogy and sociology. Special mathematics education then applies these principles to the specific domain of mathematical concepts and skills, with the aim of providing solutions for teaching mathematics.

In the mathematical or fundamental conception, didactics is presented as a science that deals with the production and communication of knowledge, focusing specifically on the unique aspects of this production and communication. The objects of study in this conception are the essential operations of the diffusion of knowledge, the conditions of this diffusion and the transformations it causes both in knowledge and in its users. In addition, this conception examines the institutions and activities that aim to facilitate these operations.

Research problems arising from the fundamental conception tend to be more theoretical in nature, often involving model building. The ultimate goal of didactics, according to this conception, is to build a theory of teaching processes that provides a practical mastery of classroom phenomena. Research in the field of Mathematics Education, like other fields such as medicine, agriculture and management, requires a combination of theoretical and practical developments. This involves studying the foundations of cognitive development and individual differences in mathematics learning, as well as addressing decision-making problems in classrooms, schools and teacher training programmes.

Research in this field covers a spectrum from pure research that may not have immediate applicability to technological research and development, to the development of educational materials that are tested and evaluated in laboratories and classrooms. Each of the different concepts within Mathematics Education is characterized by the types of problems they address.

Mathematics Didactics challenges reductionism by highlighting the limitations of general psychopedagogical theories such as behaviourism and constructivism when applied to teaching specific content. It emphasises the importance of the knowledge that is transmitted and suggests the need for content-specific theories that explain the functioning of the educational system from a knowledge-based perspective. This view is shared by Freudenthal, who expresses scepticism towards general learning theories and emphasises the uniqueness of mathematics in terms of pedagogical approaches.

The French school, still in the early stages of developing its theoretical framework, prioritizes theoretical issues over technical ones due to the lack of secure reference points for the proposals. However, considering the complexity of the teaching system, optimizing its functioning requires a collaborative effort between different research perspectives, both theoretical and applied.

The fundamental conception of Mathematics Education, with its mathematical perspective, plays a significant role in identifying theoretical concepts and didactic phenomena that contribute to the dissemination of mathematical knowledge. The connection between theory and practice and the social change that theoretical research demands require the creation of an "interface" that is currently underdeveloped. This interface could potentially be formed through explicit recognition of action research, which aims to achieve social change and empowerment. Research conducted with the active participation of teachers in research teams can serve as an interface within the teaching system. Kilpatrick advocates closer collaboration between researchers and teachers, emphasizing the need for joint efforts in research and implementation. This aligns with a sociocritical perspective of action research, which seeks to optimize the functioning of the entire system.

Research paradigms

When attempting to critically evaluate research findings in Mathematics Education, it becomes evident that their nature is relative to the specific circumstances of the participants (teachers and students) and the context in which they occur. Thus, it is noteworthy that empirical findings in mathematics education not only lack universality across different contexts, but their validity may also change over time due to the ever-changing society in which mathematics education is conducted. Another factor that significantly influences the validity and significance of research findings is the perspective from which the research is conducted, known as the research paradigm.

There are two extremes of this spectrum: the positivist or process-product approach, which aims to discover laws and confirm hypotheses about behaviors and procedures associated with student achievement, and the interpretive approach, which seeks to understand the personal meaning of events, to study the interactions between individuals and their environment, and to explore the thoughts, attitudes, and perceptions of participants.

The positivist or process-product program employs quantitative methods, often using systematic measurement, experimental designs, and mathematical modeling, whereas the interpretive program (including ecological and ethnographic approaches) is associated with naturalistic observations, case studies, ethnography, and narrative reporting. Several distinctive features are found and highlighted between these two approaches: the limited involvement of positivist researchers in the lives or activities of their subjects as compared to ethnographers, the lack of interest among positivist researchers in the intersubjective meanings that may arise in the schools or classrooms studied, the infrequent use of sociocultural theories by positivist researchers to interpret their findings, and the limited attention paid by educational anthropologists within the interpretive approach to cognitive abilities, theories of cognitive development and information processing, the reluctance to manipulate variables and force natural events, and the rare attempt to address educational problems.

These disparate programs coexist and have coexisted in the field of teaching and learning, including mathematics, particularly in research conducted from a multidisciplinary perspective. However, much of current educational research, especially the most innovative designs, can be classified as occupying an intermediate position between these paradigms. A research model is therefore proposed that comprises four dimensions or suppositional modes: deductive-inductive, generative-verifying, constructive-enumerative, and subjective-objective.

The deductive-inductive dimension refers to the reliance on existing theories or the generation of new theories through the research process. The generative-verifying dimension relates to the degree to which the results of one group can be generalized to others, and verifiable research aims to establish generalizations beyond a single group. The modes of formulation and design of variables and categories of analysis define the constructive-enumerative dimension, while the subjective-objective dimension refers to the constructs that are studied in relation to the participants involved. In addition to these paradigms, there is a third socio-critical paradigm, which advocates connecting research with practice to promote greater freedom and autonomy among participants. Mere observation of educational encounters in a classroom is insufficient; it is also necessary to provide direct guidance to practice, which requires greater collaboration between teachers and researchers.

An example of how several paradigms can be integrated is demonstrated by research conducted by the French School of Mathematics Education. This research focuses on the study of how knowledge is formed, controlled and reproduced in the school environment. An important aspect of this research is the exploration of the relationship between the two subsystems involved - knowledge and students - particularly through the problematic situation and the management of this interaction by the teacher.

The methodology employed in this research program is guided by certain assumptions, including the need for a holistic and case-study approach due to the complexity of the phenomenon under investigation, as well as the use of multiple data collection techniques. Furthermore, the specificity of mathematical knowledge allows for the generation of hypotheses from the study of this knowledge and its epistemological origins. As a result, this research program incorporates elements from different paradigms. For example, features of the positivist-experimental paradigm are evident in the careful preparation of lessons, the formulation of hypotheses based on a general theory, and the use of statistical methods for data analysis.

On the other hand, the ecological-ethnographic paradigm is reflected in the holistic and qualitative approach to the study of the phenomenon, the interest in the variables and interrelations of the process, the possibility of generating new hypotheses during the research and the use of multiple data collection techniques, including ethnographic methods such as observation. In general, the research paradigm adopted by the mathematical conception of Mathematics Education is situated between deductive and inductive reasoning, as well as between generative and enumerative approaches, combining elements from both ends of the spectrum.

Adopting a systemic perspective can help resolve any conflicts between different ideas and models. To achieve this, we need an integrative approach that considers theory, development and practice, and embraces positivism, interpretivism and critique. These different viewpoints should be seen as complementary and part of a broader understanding. According to Steiner (1985), the concept of complementarity is a useful tool for understanding the relationships between various types and levels of knowledge and activity.

Interdisciplinary and fundamental perspectives are compatible and can work together. By considering Mathematics Education as part of mathematics, we can establish a "mathematical didactics" of mathematics, similar to mathematical logic or metamathematics. However, this science cannot replace the contributions made by other sciences. Teaching situations involve multiple aspects and phenomena, and Didactics (in its fundamental sense) has not yet fully explored and explained these phenomena with specific concepts and methods.

On the other hand, the incorporation of external knowledge is crucial and must be done under the guidance of a specific theory. This approach allows for a healthy relationship between science and technique in teaching, rather than a relationship based on prescription and reproduction. Kilpatrick (1981) also advocates eclecticism with regard to methods. We should not abandon quantitative statistical techniques, which are only just beginning to be applied, in favour of exclusively ethnographic methods.

Exploratory data analysis can complement quantitative methods in the field of mathematics education. Kilpatrick also suggests that researchers should adopt a convergent approach, where studies explore a topic from multiple perspectives using various methods, rather than focusing on replication studies. In summary, the questions raised in this discussion are essential aspects of the development program proposed by Steiner (1985) for the Theory of Mathematics Education. These aspects include identifying and addressing key issues in the orientation, foundation, methodology, and organization of Mathematics Education as a discipline, and developing a comprehensive approach to Mathematics Education as a whole, considering it as an interactive system encompassing research, development, and practice, and emphasizing the dynamic role of theory-practice exchange and interdisciplinary cooperation.

The consolidation of mathematics teaching

The recognition of Mathematics Didactics as an "area of knowledge" by the Council of Universities in 1984, together with the implementation of the University Reform Law (LRU) in the same year, has paved the way for the creation of university departments dedicated to this field in Spain. These departments have played a crucial role in the advancement of mathematics education, as they are entrusted with teaching and research responsibilities in the relevant areas of knowledge.

The departments have access to important research resources, including more than 200 permanent professors dedicated to research and specific bibliographic collections. Institutional consolidation is also evidenced by the existence of doctoral programs and the defense of doctoral theses on the teaching and learning of mathematics, as well as the financing of research projects in competition with other areas of knowledge.

In 1997, the Society for Research in Mathematics Education (SEIEM) was formed, demonstrating the growing awareness of the specific interests and needs of the mathematics education research community. The realm of practical action is primarily the domain of the teacher, who is responsible for instructing one or more groups of students in mathematics. A teacher's primary goal is to enhance student learning, so his or her primary interest is to obtain information that can have an immediate impact on his or her teaching.

On the other hand, the technological component, also known as applied research, is more focused on prescribing solutions and developing action devices. This field is inhabited by curriculum designers, authors of school textbooks and creators of teaching materials. Finally, scientific research, which encompasses basic, analytical and descriptive studies, is concerned with the development of theories. This type of research is usually carried out in university institutions. Mathematics education is a complex and diverse system consisting of three distinct components or fields:

- Firstly, there is a practical and reflective action, which implies that teachers actively participate in the teaching and learning processes related to mathematics.

- Secondly, there is teaching technology, which focuses on the development of materials and resources using scientific knowledge.
- Finally, there is scientific research, which aims to understand the general functioning of mathematics teaching, as well as specific teaching systems involving teachers, students and mathematical knowledge.

Despite their shared interest in improving mathematics education, these three fields have different perspectives, goals, available resources, operational rules, and constraints. Internationally, mathematics education has also experienced a consolidation with the existence of similar research institutions and institutes in countries such as Mexico and Germany. In addition, there are several research journals and handbooks dedicated to the field, as well as international conferences that provide avenues for researchers to share their findings and collaborate. The ICMI, an international commission on mathematics instruction, has played an important role in promoting research in mathematics education throughout the 20th century. Its study conducted in Washington in 1994 highlighted the maturity of mathematics education as a scientific discipline with its own goals and methods, further solidifying its status as a distinct field of study.

In terms of research programmes and methods, there has been a shift from the use of a primarily psychostatistical approach in the 1970s and 1980s, which focused on tests and their reliability. There is now a proliferation of methods, the exploration of different research agendas and the adoption of eclectic positions. This does not mean that the psychological approach has lost importance, as demonstrated by the vitality of the international PME group.

Research is currently being conducted using a variety of approaches, including interpretive, ethnographic, anthropological, and sociocritical methods. Some argue that this diversity is beneficial, as it allows for different perspectives to be considered. However, I believe that it can lead to confusion among research communities and make efforts less productive. The multitude of approaches, theories, and methods in mathematics education research calls for a more structured and organized approach, similar to the philosophy of science.

Although mathematics education can be considered a mature discipline sociologically, it may not be so philosophically or methodologically. The problem of diversity in theories has been addressed by the European Congress of Mathematics Education (CERME) in its working group, which has led to the publication of several papers in conference proceedings and in the journal ZDM.

These challenges include difficulties in communication due to different assumptions and languages, discrepancies in empirical results due to different perspectives, and obstacles to scientific progress. It is argued that for the diversity of theories to be fruitful, different approaches and traditions must interact. To address these challenges, strategies that connect theories and theoretical approaches must be actively sought. This can be done through empirical studies that combine different theoretical approaches, developing theories as part of a connected set to reduce their number and clarify their strengths and weaknesses, and fostering a discourse on theory development and its qualities in mathematics education research, which also considers metatheoretical and methodological considerations.

When discussing the aspect of mathematics education known as reflective practice, it is important to acknowledge the important role played by mathematics teachers' associations at various levels: regional, national and international. This is evidenced by the existence of organisations such as the Spanish Federation of Associations of Mathematics Teachers, which is made up of 12 regional societies, as well as their respective journals and conferences aimed at teachers.

At the international level we see the influence of powerful institutions such as the NCTM in the USA, the ICME and the FISEM, together with its journal UNIÓN. However, it is crucial to recognise that these activities often have limited connections with the scientific and academic component of mathematics education. This is evident through the existence of independent professional societies and separate journals for "teachers" and "researchers" in countries such as Spain, France and Portugal.

This disconnect is evident in the development of mathematics curricula, which have traditionally been prepared by commissions that overlook the expertise of specialized university departments. The separation between academia and practice is most pronounced in the initial training and continuing professional development of secondary school mathematics teachers, where there is limited involvement of mathematics education specialists. In conclusion, while mathematics education has made significant progress as an academic discipline on the international stage over the past three decades, its development has been uneven in different aspects and particularly in the integration between them.

Conclusion

Freudenthal was a strong advocate of reforming traditional mathematics education. His extensive work as a founder and active participant in groups such as the International Council on Psychological and Mathematical Education (PME) and the International Commission for Research and Improvement of Mathematics Education (CIEAEM) contributed to his fame. In these forums, he expressed his opposition to the dominant pedagogical and didactic approaches of the mid-twentieth century, such as performance goal theory, structured assessment tests and standardized educational surveys, and the direct application of Piaget's structuralism and constructivism in the classroom.

Hans Freudenthal, a German-born mathematician and educator, earned his PhD at the University of Berlin. However, due to his Jewish background, he was forced to emigrate from Germany during the rise of the Nazi regime. He sought refuge in the Netherlands, where he continued his studies and developed pedagogical theories. Unfortunately, he had to go into hiding during World War II. Freudenthal believed that the learning process should be based on situations that require organization. He criticized Piaget for trying to impose psychological development on the system of categories used by mathematicians, using mathematical terms with different meanings.

Drawing on his own experience, Freudenthal argued that learning is more closely related to language development than to cognitive development. He was concerned about how Piaget's work influenced teachers who turned research findings into guidelines for mathematics education, turning an epistemological theory into a violation of pedagogical theory.

He discussed with Chevallard his theory of transposition, which he believed was based on the expert knowledge of mathematicians. Freudenthal argued that mathematics taught in schools should not reflect any interpretation of philosophical or scientific ideas unless they were much older. Freudenthal's opposition to the then-prevailing psychology, pedagogy, and teaching methods was founded. This was based on his extensive knowledge of mathematics, his passion for teaching mathematics, and his direct experience in the classroom. He questioned the artificial nature of Bloom's educational goals and fields of study, arguing that they had a negative impact on both academic and developmental tests. He accused Bloom of viewing learning as a process in which knowledge is simply transmitted into the student's head. Similarly, he disagreed with Gagne's view that learning is a continuous process, developing from simple to complex structures.

In conclusion, Freudenthal believes that learning involves sudden leaps in rethinking, demonstrated by students finding shortcuts in their strategies, changing perspectives, and using models with varying degrees of formality. However, Freudenthal's references to non-mathematical authors are limited; he acknowledges the influence of Decroli, whose interests coincided with his own theories on learning mathematics in everyday contexts, and Dewey, in whom he sees similarities in the idea of guided rethinking and was influenced by Lagenveld's phenomenological pedagogy.

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